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HARMONIC ANALYSIS ASSOCIATED WITH A GENERALIZED BESSEL-STRUVE OPERATOR ON THE REAL LINE

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Abstract. In this paper we consider a generalized Bessel-Struve operator $l_{\alpha,n}$ on the real line, which generalizes the Bessel-Struve operator l_α, we define the generalized Bessel-Struve intertwining operator which turn out to be transmutation operator between $l_{\alpha,n}$ and the second derivative operator $\frac{d^2}{dx^2}$. We build the generalized Weyl integral transform and we establish an inversion theorem of the generalized Weyl integral transform. We exploit the generalized Bessel-Struve intertwining operator and the generalized Weyl integral transform, firstly to develop a new harmonic analysis on the real line corresponding to $l_{\alpha,n}$, and secondly to introduce and study the generalized Sonine integral transform $S^{n,m}_{\alpha,\beta}$. We prove that $S^{n,m}_{\alpha,\beta}$ is a transmutation operator from $l_{\alpha,n}$ to $l_{\beta,n}$. As a side result we prove Paley-Wiener theorem for the generalized Bessel-Struve transform associated with the generalized Bessel-Struve operator.

I. Introduction

In this paper we consider the generalized Bessel-Struve oprator $l_{\alpha,n}$, $\alpha > \frac{-1}{2}$, defined on $\mathbb{R}$ by

$$(1)\quad l_{\alpha,n}u(x) = \frac{d^2u}{dx^2}(x) + \frac{2\alpha+1}{x}\frac{du}{dx}(x) - \frac{4n(\alpha+n)}{x^2}u(x) - \frac{(2\alpha+4n+1)}{x}D(u)(0)$$

where $D = x^{2n}\frac{d}{dx} \circ x^{-2n}$ and $n = 0, 1, \ldots$. For $n = 0$, we regain the Bessel-Struve operator

$$(2)\qquad l_\alpha u(x) = \frac{d^2u}{dx^2}(x) + \frac{2\alpha+1}{x}\left[\frac{du}{dx}(x) - \frac{du}{dx}(0)\right].$$

Through this paper, we provide a new harmonic analysis on the real line corresponding to the generalized Bessel-Struve operator $l_{\alpha,n}$.
The outline of the content of this paper is as follows.
Section 2 is dedicated to some properties and results concerning the Bessel-Struve transform.

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In section 3, we construct a pair of transmutation operators $\mathcal{X}_{\alpha,n}$ and $W_{\alpha,n}$, afterwards we exploit these transmutation operators to build a new harmonic analysis on the real line corresponding to operator $l_{\alpha,n}$.

II. Preliminaries

Throughout this paper assume $\alpha > \beta > \frac{-1}{2}$. We denote by

- $E(\mathbb{R})$ the space of C^∞ functions on $\mathbb{R}$, provided with the topology of compact convergence for all derivatives. That is the topology defined by the semi-norms

$$p_{a,m}(f) = sup_{x\in[-a,a]}|f^{(k)}(x)|,\ a > 0,\ m \in \mathbb{N},\ and\ 0 \leq k \leq m.$$

- $D_a(\mathbb{R})$, the space of C^∞ functions on $\mathbb{R}$, which are supported in $[-a, a]$, equipped with the topology induced by $E(\mathbb{R})$.
- $D(\mathbb{R}) = \bigcup_{a>0} D_a(\mathbb{R})$, endowed with inductive limit topology.
- $L^p_\alpha(\mathbb{R})$ the class of measurable functions f on $\mathbb{R}$ for which $\|f\|_{p,\alpha} < \infty$, where

$$\|f\|_{p,\alpha} = \left(\int_{\mathbb{R}} |f(x)|^p |x|^{2\alpha+1} dx\right)^{\frac{1}{p}}, \quad ifp < \infty,$$

  and $\|f\|_{\infty,\alpha} = \|f\|_\infty = ess\ sup_{x\geq 0}|f(x)|$.
- $\dfrac{d}{dx^2} = \dfrac{1}{2x}\dfrac{d}{dx}$, where $\dfrac{d}{dx}$ is the first derivative operator.

In this section we recall some facts about harmonic analysis related to the Bessel-Struve operator l_α. We cite here, as briefly as possible, only some properties. For more details we refer to [2, 3].
For $\lambda \in \mathbb{C}$, the differential equation:

$$(3) \qquad \begin{cases} l_\alpha u(x) = \lambda^2 u(x) \\ u(0) = 1,\ u'(0) = \frac{\lambda\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{3}{2})} \end{cases}$$

possesses a unique solution denoted $\Phi_\alpha(\lambda x)$. This eigenfunction, called the Bessel-Struve kernel, is given by:

$$\Phi_\alpha(\lambda x) = j_\alpha(i\lambda x) - ih_\alpha(i\lambda x), \quad x \in \mathbb{R}.$$

j_α and h_α are respectively the normalized Bessel and Struve functions of index α .These kernels are given as follows

$$j_\alpha(z) = \Gamma(\alpha+1)\sum_{k=0}^{+\infty} \frac{(-1)^k \left(\frac{z}{2}\right)^{2k}}{k!\Gamma(k+\alpha+1)}$$

and

$$h_\alpha(z) = \Gamma(\alpha+1)\sum_{k=0}^{+\infty} \frac{(-1)^k \left(\frac{z}{2}\right)^{2k+1}}{\Gamma\left(k+\frac{3}{2}\right)\Gamma\left(k+\alpha+\frac{3}{2}\right)}.$$

The kernel Φ_α possesses the following integral representation:

$$\Phi_\alpha(\lambda x) = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_0^1 (1-t^2)^{\alpha-\frac{1}{2}} e^{\lambda x t} dt, \quad \forall x \in \mathbb{R}, \quad \forall \lambda \in \mathbb{C}. \tag{4}$$

The Bessel-Struve intertwining operator on $\mathbb{R}$ denoted $\mathcal{X}_\alpha$ introduced by L. Kamoun and M. Sifi in [3], is defined by:

$$\mathcal{X}_\alpha(f)(x) = a_\alpha \int_0^1 (1-t^2)^{\alpha-1} f(xt)dt \ , f \in E(\mathbb{R}), \ x \in \mathbb{R}, \tag{5}$$

where

$$a_\alpha = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}. \tag{6}$$

The Bessel-Struve kernel Φ_α is related to the exponential function by

$$\forall x \in \mathbb{R}, \quad \forall \lambda \in \mathbb{C}, \quad \Phi_\alpha(\lambda x) = \mathcal{X}_\alpha(e^{\lambda \cdot})(x). \tag{7}$$

$\mathcal{X}_\alpha$ is a transmutation operator from l_α into $\frac{d^2}{dx^2}$ and verifies

$$l_\alpha \circ \mathcal{X}_\alpha = \mathcal{X}_\alpha \circ \frac{d^2}{dx^2}. \tag{8}$$

Theorem 1. *The operator $\mathcal{X}_\alpha$, $\alpha > \frac{-1}{2}$ is topological isomorphism from $E(\mathbb{R})$ onto itself. The inverse operator $\mathcal{X}_\alpha^{-1}$ is given for all $f \in E(\mathbb{R})$ by*

(i) *if $\alpha = r + k$, $k \in \mathbb{N}$, $\frac{-1}{2} < r < \frac{1}{2}$*

$$\mathcal{X}_\alpha^{-1}(f)(x) = \frac{2\sqrt{\pi}}{\Gamma(\alpha+1)\Gamma(\frac{1}{2}-r)} x(\frac{d}{dx^2})^{k+1}\left[\int_0^x (x^2-t^2)^{-r-\frac{1}{2}} f(t)|t|^{2\alpha+1}dt\right]. \tag{9}$$

(ii) *if $\alpha = \frac{1}{2} + k$, $k \in \mathbb{N}$*

$$\mathcal{X}_\alpha^{-1}(f)(x) = \frac{2^{2k+1}k!}{(2k+1)!} x(\frac{d}{dx^2})^{k+1}(x^{2k+1}f(x)), \ x \in \mathbb{R}. \tag{10}$$

Definition 1. *The Bessel-Struve transform is defined on $L^1_\alpha(\mathbb{R})$ by*

$$\forall \lambda \in \mathbb{R}, \quad \mathcal{F}^\alpha_{B,S}(f)(\lambda) = \int_{\mathbb{R}} f(x)\Phi_\alpha(-i\lambda x)|x|^{2\alpha+1}dx. \tag{11}$$

Proposition 1. *If $f \in L^1_\alpha(\mathbb{R})$ then $\|\mathcal{F}^\alpha_{B,S}(f)\|_\infty \le \|f\|_{1,\alpha}$.*

Theorem 2. *(Paley-Wiener) Let $a > 0$ and f a function in $\mathcal{D}_a(\mathbb{R})$ then $\mathcal{F}^\alpha_{B,S}$ can be extended to an analytic function on $\mathbb{C}$ that we denote again $\mathcal{F}^\alpha_{B,S}(f)$ verifying*

$$\forall k \in \mathbb{N}^*, \quad |\mathcal{F}^\alpha_{B,S}(f)(z)| \le Ce^{a|z|}.$$

Definition 2. *For $f \in L^1_\alpha(\mathbb{R})$ with bounded support, the integral transform W_α, given by*

$$W_\alpha(f)(x) = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_{|x|}^{+\infty} (y^2-x^2)^{\alpha-\frac{1}{2}} y f(sgn(x)y)dy, \quad x \in \mathbb{R}\backslash\{0\} \tag{12}$$

is called Weyl integral transform associated with Bessel-Struve operator.

Proposition 2. (i) W_α *is a bounded operator from* $L^1_\alpha(\mathbb{R})$ *to* $L^1(\mathbb{R})$, *where* $L^1(\mathbb{R})$ *is the space of lebesgue-integrable functions.*

(ii) *Let* f *be a function in* $E(\mathbb{R})$ *and* g *a function in* $L_\alpha(\mathbb{R})$ *with bounded support, the operators* $\mathcal{X}_\alpha$ *and* W_α *are related by the following relation*

$$\int_{\mathbb{R}} \mathcal{X}_\alpha(f)(x)g(x)|x|^{2\alpha+1}dx = \int_{\mathbb{R}} f(x)W_\alpha(g)(x)dx. \tag{13}$$

(iii) $\forall f \in L^1_\alpha(\mathbb{R})$, $\mathcal{F}^\alpha_{B,S} = \mathcal{F} \circ W_\alpha(f)$ *where* $\mathcal{F}$ *is the classical Fourier transform defined on* $L^1(\mathbb{R})$ *by*

$$\mathcal{F}(g)(\lambda) = \int_{\mathbb{R}} g(x)e^{-i\lambda x}dx.$$

We designate by K_0 the space of functions f infinitely differentiable on $\mathbb{R}^*$ with bounded support verifying for all $n \in \mathbb{N}$,

$$\lim_{y\to 0^-} y^n f^{(n)}(y) \quad and \quad \lim_{y\to 0^+} y^n f^{(n)}(y)$$

exist.

Definition 3. *We define the operator* V_α *on* K_0 *as follows*

- *If* $\alpha = k + \frac{1}{2}$, $k \in \mathbb{N}$

$$V_\alpha(f)(x) = (-1)^{k+1}\frac{2^{2k+1}k!}{(2k+1)!}(\frac{d}{dx^2})^{k+1}(f(x)), \quad x \in \mathbb{R}^*.$$

- *If* $\alpha = k + r$, $k \in \mathbb{N}$, $\frac{-1}{2} < r < \frac{1}{2}$ *and* $f \in K_0$

$$V_\alpha(f)(x) = \frac{(-1)^{k+1}2\sqrt{\pi}}{\Gamma(\alpha+1)\Gamma(\frac{1}{2}-r)}\left[\int_{|x|}^\infty (y^2-x^2)^{-r-\frac{1}{2}}(\frac{d}{dy^2})^{k+1}f(sgn(x)y)ydy\right], \ x \in \mathbb{R}^*.$$

Proposition 3. *Let* $f \in K_0$ *and* $g \in E(\mathbb{R})$,

- *the operators* V_α *and* $\mathcal{X}_\alpha^{-1}$ *are related by the following relation*

$$\int_{\mathbb{R}} V_\alpha(f)(x)g(x)|x|^{2\alpha+1}dx = \int_{\mathbb{R}} f(x)\mathcal{X}_\alpha^{-1}(g)(x)dx. \tag{14}$$

- V_α *and* W_α *are related by the following relation*

$$V_\alpha(W_\alpha(f)) = W_\alpha(V_\alpha(f)) = f. \tag{15}$$

Definition 4. *Let* f *be a continuous function on* $\mathbb{R}$. *We define the Sonine integral transform as in* [4] *by, for all* $x \in \mathbb{R}$

$$S_{\alpha,\beta}(f)(x) = c(\alpha,\beta)\int_0^1 (1-r^2)^{\alpha-\beta-1}f(rx)r^{2\beta+1}dr, \tag{16}$$

where

$$c(\alpha,\beta) = \frac{2\Gamma(\alpha+1)}{\Gamma(\beta+1)\Gamma(\alpha-\beta)}. \tag{17}$$

Proposition 4. (i) *The classical Sonine integral formula may be formulated as follows*

$$\Phi_\alpha(\lambda x) = c(\alpha,\beta)\int_0^1 (1-t^2)^{\alpha-\beta-1}\Phi_\beta(\lambda t x)t^{2\beta+1}dt. \tag{18}$$

(ii) *The Sonine integral transform verifies*

$$S_{\alpha,\beta}(\Phi_\beta(\lambda.))(x) = \Phi_\alpha(\lambda x), \quad x \in \mathbb{R}. \tag{19}$$

(iii) *For f a function of class C^2 on $\mathbb{R}$, $S_{\alpha,\beta}(f)$ is a function of class C^2 on $\mathbb{R}$ and*

$$\forall x \in \mathbb{R},\ l_\alpha(S_{\alpha,\beta}(f))(x) = S_{\alpha,\beta}(l_\beta(f))(x). \tag{20}$$

(iv) *The Sonine integral transform is a topological isomorphism from $E(\mathbb{R})$ onto itself. Furthermore, it verifies*

$$S_{\alpha,\beta} = \mathcal{X}_\alpha \circ \mathcal{X}_\beta^{-1}. \tag{21}$$

(v) *The inverse operator is*

$$S_{\alpha,\beta}^{-1} = \mathcal{X}_\beta \circ \mathcal{X}_\alpha^{-1}. \tag{22}$$

Definition 5. *For f continuous function on $\mathbb{R}$, with compact support, we define the Dual Sonine transform denoted ${}^tS_{\alpha,\beta}$ by*

$${}^tS_{\alpha,\beta}(f)(x) = c(\alpha,\beta)\int_{|x|}^\infty (y^2-x^2)^{\alpha-\beta-1}yf(sgn(x)y)dy, \quad x \in \mathbb{R}^*.$$

Theorem 3. *The dual Sonine transform verifies the following relations for all $f \in D(\mathbb{R})$ and $g \in E(\mathbb{R})$, we have*

(i)
$$\int_{\mathbb{R}} S_{\alpha,\beta}(g)(x)f(x)|x|^{2\alpha+1}dx = \int_{\mathbb{R}} {}^tS_{\alpha,\beta}(f)(x)g(x)|x|^{2\beta+1}dx.$$

(ii)
$${}^tS_{\alpha,\beta}(f) = V_\beta(W_\alpha(f)).$$

(iii)
$$\mathcal{F}_{B,S}^\beta(f) = \mathcal{F}_{B,S}^\alpha \circ^t S_{\alpha,\beta}(f).$$

III. Harmonic analysis associated with $l_{\alpha,n}$

Throughout this section assume $\alpha > \beta > \frac{-1}{2}$ and $n = 0,1,2,...$. We denoted by

- $\mathcal{M}_n$ the map defined by $\mathcal{M}_n f(x) = x^{2n}f(x)$.
- $E_n(\mathbb{R})$ (resp $D_n(\mathbb{R})$) stand for the subspace of $E(\mathbb{R})$ (resp. $D(\mathbb{R})$) consisting of functions f such that

$$f(0) = ... = f^{(2n-1)}(0) = 0.$$

- $D_{a,n}(\mathbb{R}) = D_a(\mathbb{R}) \cap E_n(\mathbb{R})$ where $a > 0$.

- $L^p_{\alpha,n}(\mathbb{R})$ the class of measurable functions f on $\mathbb{R}$ for which

$$\|f\|_{p,\alpha,n} = \|\mathcal{M}_n^{-1} f\|_{p,\alpha+2n} < \infty.$$

i. **Transmutation operators.**

For $\lambda \in \mathbb{C}$ and $x \in \mathbb{R}$, put

$$\Psi_{\lambda,\alpha,n}(x) = x^{2n}\Phi_{\alpha+2n}(\lambda x) \tag{23}$$

where $\Phi_{\alpha+2n}$ is the Bessel-Struve kernel of index $\alpha + 2n$.

Lemma 1. *(i) The map $\mathcal{M}_n$ is a topological isomorphism*

- *from $E(\mathbb{R})$ onto $E_n(\mathbb{R})$.*
- *from $D(\mathbb{R})$ onto $D_n(\mathbb{R})$.*

(ii) For all $f \in E(\mathbb{R})$

$$l_{\alpha,n} \circ \mathcal{M}_n(f) = \mathcal{M}_n \circ l_{\alpha+2n}(f). \tag{24}$$

Proof. Assertion (i) is easily checked (see [1]).
By (1) and (2) we have for any $f \in E(\mathbb{R})$,

$$\begin{aligned} l_{\alpha,n}(x^{2n}f)(x) &= (x^{2n}f)'' + \frac{2\alpha+1}{x}(x^{2n}f)' - \frac{4n(\alpha+n)}{x^2}(x^{2n}f(x)) - (2\alpha+4n+1)x^{2n-1}f'(0) \\ &= x^{2n}\left(f''(x) - \frac{2\alpha+4n+1}{x}(f'(x) - f'(0))\right) \\ &= x^{2n} l_{\alpha+2n} f(x). \end{aligned}$$

which proves Assertion (ii). ■

Proposition 5. *(i) $\Psi_{\lambda,\alpha,n}$ satisfies the differential equation*

$$l_{\alpha,n}\Psi_{\lambda,\alpha,n} = \lambda^2 \Psi_{\lambda,\alpha,n}.$$

(ii) $\Psi_{\lambda,\alpha,n}$ possesses the following integral representation:

$$\Psi_{\lambda,\alpha,n}(x) = \frac{2\Gamma(\alpha+2n+1)}{\sqrt{\pi}\Gamma(\alpha+2n+\frac{1}{2})} x^{2n} \int_0^1 (1-t^2)^{\alpha+2n-\frac{1}{2}} e^{\lambda x t} dt, \quad \forall x \in \mathbb{R}, \quad \forall \lambda \in \mathbb{C}.$$

Proof.
By (23)

$$\Psi_{\lambda,\alpha,n} = \mathcal{M}_n(\Phi_{\alpha+2n}(\lambda x)),$$

using (3) and (24) we obtain

$$\begin{aligned} l_{\alpha,n}(\Psi_{\lambda,\alpha,n}) &= l_{\alpha,n} \circ \mathcal{M}_n(\Phi_{\alpha+2n}(\lambda .)) \\ &= \mathcal{M}_n \circ l_{\alpha+2n}(\Phi_{\alpha+2n}(\lambda .)) \\ &= \lambda^2 \Psi_{\lambda,\alpha,n}, \end{aligned}$$

which proves (i). Statement (ii) follows from (4) and (23). ■

Definition 6. *For $f \in E(\mathbb{R})$, we define the generalized Bessel-Struve intertwining operator $\mathcal{X}_{\alpha,n}$ by*

$$\mathcal{X}_{\alpha,n}(f)(x) = a_{\alpha+2n}x^{2n}\int_0^1 (1-t^2)^{\alpha+2n-1}f(xt)dt \ , f \in E(\mathbb{R}),\ x \in \mathbb{R}$$

where $a_{\alpha+2n}$ is given by (6).

Remark 1.
- *For $n = 0$, $\mathcal{X}_{\alpha,n}$ reduces to the Bessel-Struve intertwining operator.*
- *It is easily checked that*

(25)
$$\mathcal{X}_{\alpha,n} = \mathcal{M}_n \circ \mathcal{X}_{\alpha+2n}.$$

- *Due to (7), (23) and (25) we have*

$$\Psi_{\lambda,\alpha,n}(x) = \mathcal{X}_{\alpha,n}(e^{\lambda \cdot})(x).$$

Proposition 6. *$\mathcal{X}_{\alpha,n}$ is a transmutation operator from $l_{\alpha,n}$ into $\frac{d^2}{dx^2}$ and verifies*

$$l_{\alpha,n} \circ \mathcal{X}_{\alpha,n} = \mathcal{X}_{\alpha,n} \circ \frac{d^2}{dx^2}.$$

Proof. It follows from (8), (25) and lemma 1 (ii) that

$$\begin{aligned} l_{\alpha,n} \circ \mathcal{X}_{\alpha,n} &= l_{\alpha,n} \circ \mathcal{M}_n\mathcal{X}_{\alpha+2n} \\ &= \mathcal{M}_n \circ l_{\alpha+2n}\mathcal{X}_{\alpha+2n} \\ &= \mathcal{M}_n\mathcal{X}_{\alpha+2n} \circ \frac{d^2}{dx^2} \\ &= \mathcal{X}_{\alpha,n} \circ \frac{d^2}{dx^2}. \end{aligned}$$

■

Theorem 4. *The operator $\mathcal{X}_{\alpha,n}$ is an isomorphism from $E(\mathbb{R})$ onto $E_n(\mathbb{R})$. The inverse operator $\mathcal{X}_{\alpha,n}^{-1}$ is given for all $f \in E_n(\mathbb{R})$ by*

(i) *if $\alpha = r + k$, $k \in \mathbb{N}$, $\frac{-1}{2} < r < \frac{1}{2}$*

$$\mathcal{X}_{\alpha,n}^{-1}f(x) = \frac{2\sqrt{\pi}}{\Gamma(\alpha+2n+1)\Gamma(\frac{1}{2}-r)}x(\frac{d}{dx^2})^{k+2n+1}\left[\int_0^x (x^2-t^2)^{-r-\frac{1}{2}}f(t)|t|^{2\alpha+2n+1}dt\right].$$

(ii) *if $\alpha = \frac{1}{2} + k$, $k \in \mathbb{N}$*

$$\mathcal{X}_{\alpha,n}^{-1}f(x) = \frac{2^{2k+4n+1}(k+2n)!}{(2k+4n+1)!}x(\frac{d}{dx^2})^{k+2n+1}(x^{2k+2n+1}f(x)),\ x \in \mathbb{R}.$$

Proof. A combination of (25), Lemma 1 and Theorem 1 shows that $\mathcal{X}_{\alpha,n}$ is an isomorphism from $E(\mathbb{R})$ onto $E_n(\mathbb{R})$. Let $\mathcal{X}_{\alpha,n}^{-1}$ the inverse operator of $\mathcal{X}_{\alpha,n}$, we have

$$\mathcal{X}_{\alpha,n}^{-1}(f) = (\mathcal{X}_{\alpha,n}(f))^{-1}.$$

Using (25) we can deduce that

$$\mathcal{X}_{\alpha,n}^{-1}(f) = (\mathcal{M}_n\mathcal{X}_{\alpha+2n}(f))^{-1}$$

$$\mathcal{X}_{\alpha,n}^{-1}(f) = \mathcal{X}_{\alpha+2n}^{-1}\mathcal{M}_n^{-1}(f). \tag{26}$$

By (9) and (10) we obtain the desired result.
■

ii. **The generalized Weyl integral transform.**

Definition 7. *For $f \in L^1_{\alpha,n}(\mathbb{R})$ with bounded support, the integral transform $W_{\alpha,n}$, given by*

$$W_{\alpha,n}(f(x)) = a_{\alpha+2n}\int_{|x|}^{+\infty}(y^2-x^2)^{\alpha+2n-\frac{1}{2}}y^{1-2n}f(sgn(x)y)dy, \quad x\in\mathbb{R}\backslash\{0\}$$

is called the generalized Weyl integral transform associated with Bessel-Struve operator.

Remark 2.
- *By a change of variable, $W_{\alpha,n}f$ can be written*

$$W_{\alpha,n}f(x) = a_{\alpha+2n}|x|^{2\alpha+2n+1}\int_1^{+\infty}\left(t^2-1\right)^{\alpha+2n-\frac{1}{2}}t^{1-2n}f(tx)\,dt, \qquad x\in\mathbb{R}\diagdown\{0\}.$$

- *It is easily checked that*

$$W_{\alpha,n} = W_{\alpha+2n}\circ\mathcal{M}_n^{-1}. \tag{27}$$

Proposition 7. *$W_{\alpha,n}$ is a bounded operator from $L^1_{\alpha,n}(\mathbb{R})$ to $L^1(\mathbb{R})$, where $L^1(\mathbb{R})$ is the space of lebesgue-integrable.*

Proof. Let $f\in L^1_{\alpha,n}(\mathbb{R})$, by Proposition 2 (i) we can find a positif constant C such that

$$\begin{aligned}\|W_{\alpha+2n}(\mathcal{M}_n^{-1}f)\|_1 &\leq C\|\mathcal{M}_n^{-1}f\|_{1,\alpha+2n}\\ \|W_{\alpha,n}(f)\|_1 &\leq C\|f\|_{1,\alpha,n}.\end{aligned}$$

By (27) we obtain the desired result. ■

Proposition 8. *Let f be a function in $E(\mathbb{R})$ and g a function in $L^1_{\alpha,n}(\mathbb{R})$ with bounded support, the operators $\mathcal{X}_{\alpha,n}$ and $W_{\alpha,n}$ are related by the following relation*

$$\int_{\mathbb{R}}\mathcal{X}_{\alpha,n}(f)(x)g(x)|x|^{2\alpha+1}dx = \int_{\mathbb{R}}f(x)W_{\alpha,n}(g)(x)dx.$$

Proof. Using (25), (27) and Proposition 2 (ii) we obtain

$$\begin{aligned}\int_{\mathbb{R}} \mathcal{X}_{\alpha,n}(f(x))g(x)|x|^{2\alpha+1}dx &= \int_{\mathbb{R}} \mathcal{M}_n\mathcal{X}_{\alpha+2n}(f)(x)g(x)|x|^{2\alpha+1}dx \\ &= \int_{\mathbb{R}} x^{2n}\mathcal{X}_{\alpha+2n}(f)(x)g(x)|x|^{2\alpha+1}dx \\ &= \int_{\mathbb{R}} \mathcal{X}_{\alpha+2n}f(x)\frac{g(x)}{x^{2n}}|x|^{2\alpha+4n+1}dx \\ &= \int_{\mathbb{R}} f(x)W_{\alpha+2n}(\frac{g(x)}{x^{2n}})dx \\ &= \int_{\mathbb{R}} f(x)W_{\alpha,n}(g)(x)dx.\end{aligned}$$

■

Definition 8. *We define the operator $V_{\alpha,n}$ on K_0 as follows*

- *If $\alpha = k+\frac{1}{2}$, $k\in\mathbb{N}$ and $f\in K_0$*

$$V_{\alpha,n}f(x) = (-1)^{k+1}\frac{2^{2k+4n+1}(k+2n)!}{(2k+4n+1)!}x^{2n}(\frac{d}{dx^2})^{k+2n+1}(f(x)), \quad x\in\mathbb{R}^*.$$

- *If $\alpha = k+r$, $k\in\mathbb{N}$, $\frac{-1}{2}<r<\frac{1}{2}$*

$$V_{\alpha,n}f(x) = \frac{(-1)^{k+1}2\sqrt{\pi}}{\Gamma(\alpha+2n+1)\Gamma(\frac{1}{2}-r)}x^{2n}\left[\int_{|x|}^{\infty}(y^2-x^2)^{-r-\frac{1}{2}}(\frac{d}{dy^2})^{k+2n+1}f(sgn(x)y)ydy\right],\ x\in\mathbb{R}^*.$$

Remark 3. *It is easily checked that*

$$V_{\alpha,n} = \mathcal{M}_n\circ V_{\alpha+2n}. \tag{28}$$

Proposition 9. *Let $f\in K_0$ and $g\in E_n(\mathbb{R})$, the operators $V_{\alpha,n}$ and $\mathcal{X}_{\alpha,n}^{-1}$ are related by the following relation*

$$\int_{\mathbb{R}} V_{\alpha,n}f(x)g(x)|x|^{2\alpha+1}dx = \int_{\mathbb{R}} f(x)\mathcal{X}_{\alpha,n}^{-1}g(x)dx.$$

Proof. A combination of (14), (26) and (28) shows that

$$\begin{aligned}\int_{\mathbb{R}} V_{\alpha,n}(f(x))g(x)|x|^{2\alpha+1}dx &= \int_{\mathbb{R}} \mathcal{M}_nV_{\alpha+2n}(f(x))g(x)|x|^{2\alpha+1}dx \\ &= \int_{\mathbb{R}} x^{2n}V_{\alpha+2n}f(x)g(x)|x|^{2\alpha+1}dx \\ &= \int_{\mathbb{R}} V_{\alpha+2n}f(x)\frac{g(x)}{x^{2n}}|x|^{2\alpha+4n+1}dx \\ &= \int_{\mathbb{R}} f(x)\mathcal{X}_{\alpha+2n}^{-1}(\frac{g(x)}{x^{2n}})dx \\ &= \int_{\mathbb{R}} f(x)\mathcal{X}_{\alpha,n}^{-1}(g(x))dx.\end{aligned}$$

■

Theorem 5. *Let $f \in K_0$, $V_{\alpha,n}$ and $W_{\alpha,n}$ are related by the following relation*

$$V_{\alpha,n}(W_{\alpha,n}(f)) = W_{\alpha,n}(V_{\alpha,n}(f)) = f.$$

Proof. *The result follows directly from Proposition 3.(15), (27) and (28).* ∎

iii. **The generalized Sonine integral transform.**

Definition 9. *Let $f \in E_m(\mathbb{R})$. We define the generalized Sonine integral transform by, for all $x \in \mathbb{R}$*

(29)

$$S^{n,m}_{\alpha,\beta}(f)(x) = c(\alpha+2n, \beta+2m)x^{2(n-m)} \int_0^1 (1-r^2)^{\alpha-\beta+2(n-m)-1} f(rx) r^{2\beta+2m+1} dr,$$

where $\alpha > \beta > \frac{-1}{2}$ and m, n two non-negative integers such that $n \geq m$. For $n = m = 0$, $S^{n,m}_{\alpha,\beta}$ reduces to the classical Sonine integral transform $S_{\alpha,\beta}$.

Remark 4. *Due to (16) and (29)*

$$S^{n,m}_{\alpha,\beta} = \mathcal{M}_n \circ S_{\alpha+2n,\beta+2n} \circ \mathcal{M}_m^{-1}. \tag{30}$$

In the next Proposition, we establish an analogue of Sonine formula

Proposition 10. *We have the following relation*

(31)

$$\Psi_{\lambda,\alpha,n}(x) = c(\alpha+2n, \beta+2m)x^{2(n-m)} \int_0^1 (1-t^2)^{\alpha-\beta+2(n-m)-1} \Psi_{\lambda,\beta,m}(tx) t^{2\beta+2m+1} dt.$$

Proof. A combination of (18) and (23) leads to the desired result. ∎

Remark 5. *The following relation yields from relation (31)*

$$S^{n,m}_{\alpha,\beta}(\Psi_{\lambda,\beta,m}(.))(x) = \Psi_{\lambda,\alpha,n}(x).$$

Theorem 6. *The generalized Sonine integral transform $S^{n,m}_{\alpha,\beta}(f)$ is an isomorphism from $E_m(\mathbb{R})$ onto $E_n(\mathbb{R})$ satisfying the intertwining relation*

$$l_{\alpha,n}(S^{n,m}_{\alpha,\beta}(f))(x) = S^{n,m}_{\alpha,\beta}(l_{\beta,m}(f))(x).$$

Proof. An easily combination of (20), (24), (30), Lemma 1.(i) and Proposition 4 (iv) yields $S^{n,m}_{\alpha,\beta}(f)$ is an isomorphism from $E_m(\mathbb{R})$ onto $E_n(\mathbb{R})$ and

$$\begin{aligned} l_{\alpha,n}(S^{n,m}_{\alpha,\beta}(f))(x) &= l_{\alpha,n}\mathcal{M}_n \circ S_{\alpha+2n,\beta+2m} \circ \mathcal{M}_m^{-1}(f)(x) \\ &= \mathcal{M}_n l_{\alpha+2n}(S_{\alpha+2n,\beta+2m}) \circ \mathcal{M}_m^{-1}(f)(x) \\ &= \mathcal{M}_n S_{\alpha+2n,\beta+2m} l_{\beta+2m} \circ \mathcal{M}_m^{-1}(f)(x) \\ &= \mathcal{M}_n S_{\alpha+2n,\beta+2m} \circ \mathcal{M}_m^{-1} l_{\beta,m}(f)(x) \\ &= S^{n,m}_{\alpha,\beta}(l_{\beta,m}(f))(x). \end{aligned}$$

∎

Theorem 7. *The generalized Sonine transform is a topological isomorphism from $E_m(\mathbb{R})$ onto $E_n(\mathbb{R})$. Furthermore, it verifies*

$$S^{n,m}_{\alpha,\beta} = \mathcal{X}_{\alpha,n} \circ \mathcal{X}^{-1}_{\beta,m}$$

the inverse operator is

$$(S^{n,m}_{\alpha,\beta})^{-1} = \mathcal{X}_{\beta,m} \circ \mathcal{X}^{-1}_{\alpha,n}.$$

Proof. It follows from (25), (30), Lemma 1.(i) and Proposition 4 ((iv)-(v)) that $S^{n,m}_{\alpha,\beta}$ is a topological isomorphism from $E_m(\mathbb{R})$ onto $E_n(\mathbb{R})$ and

$$\begin{aligned} S^{n,m}_{\alpha,\beta}(f) &= \mathcal{M}_n \circ S_{\alpha+2n,\beta+2m} \circ \mathcal{M}^{-1}_m(f) \\ &= \mathcal{M}_n \mathcal{X}_{\alpha+2n} \circ \mathcal{X}^{-1}_{\beta+2m} \mathcal{M}^{-1}_m(f) \\ &= \mathcal{X}_{\alpha,n} \circ \mathcal{X}^{-1}_{\beta,m}(f). \end{aligned}$$

For the inverse operator it is easily checked that

$$(S^{n,m}_{\alpha,\beta})^{-1} = \mathcal{X}_{\beta,m} \circ \mathcal{X}^{-1}_{\alpha,n}.$$

■

Definition 10. *For $f \in D_n(\mathbb{R})$ we define the dual generalized Sonine transform denoted ${}^tS_{\alpha,\beta}$ by*
(32)

$${}^tS^{n,m}_{\alpha,\beta}(f)(x) = c(\alpha+2n, \beta+2m)x^{2m} \int_{|x|}^{\infty} (y^2 - x^2)^{\alpha-\beta+2(n-m)-1} y^{1-2n} f(sgn(x)y)dy,$$

where $x \in \mathbb{R}^*$.

Remark 6. *Due to (32) and Definition 5 we have*

(33)
$${}^tS^{n,m}_{\alpha,\beta} = \mathcal{M}_m \, {}^tS_{\alpha+2n,\beta+2m} \mathcal{M}^{-1}_n.$$

Proposition 11. *The dual generalized Sonine transform verifies the following relation for all $f \in D_n(\mathbb{R})$ and $g \in E_m(\mathbb{R})$,*

$$\int_{\mathbb{R}} S^{n,m}_{\alpha,\beta} g(x) f(x) |x|^{2\alpha+1} dx = \int_{\mathbb{R}} {}^tS^{n,m}_{\alpha,\beta}(f)(x) g(x) |x|^{2\beta+1} dx.$$

Proof. A combination of (30), (33) and Theorem 3.(i) we get

$$\begin{aligned} \int_{\mathbb{R}} S^{n,m}_{\alpha,\beta}(g)(x) f(x) |x|^{2\alpha+1} dx &= \int_{\mathbb{R}} \mathcal{M}_n \circ S_{\alpha+2n,\beta+2m} \circ \mathcal{M}^{-1}_m(g)(x) f(x) |x|^{2\alpha+1} dx \\ &= \int_{\mathbb{R}} S_{\alpha+2n,\beta+2m} \circ \mathcal{M}^{-1}_m(g(x)) \mathcal{M}^{-1}_n(f)(x) |x|^{2(\alpha+2n)+1} dx \\ &= \int_{\mathbb{R}} \mathcal{M}^{-1}_m(g(x))^t S_{\alpha+2n,\beta+2m} \circ \mathcal{M}^{-1}_n f(x) |x|^{2(\beta+2m)+1} dx \\ &= \int_{\mathbb{R}} \mathcal{M}_m(g(x))^t S_{\alpha+2n,\beta+2m} \circ \mathcal{M}^{-1}_n(f)(x) |x|^{2\beta+1} dx \\ &= \int_{\mathbb{R}} g(x)^t S^{n,m}_{\alpha,\beta}(f)(x) |x|^{2\beta+1} dx. \end{aligned}$$

■

Theorem 8. *For all $f \in D_n(\mathbb{R})$, we have*

$$ {}^tS^{n,m}_{\alpha,\beta}(f) = V_{\beta,m}(W_{\alpha,n}(f)). $$

Proof. By (27), (28), (33) and Theorem 3 (ii), we get

$$ \begin{array}{rcl} {}^tS^{n,m}_{\alpha,\beta}(f) & = & \mathcal{M}_m \, {}^tS_{\alpha+2n,\beta+2m}\mathcal{M}_n^{-1}(f) \\ & = & \mathcal{M}_m V_{\beta+2m}(W_{\alpha+2n}\mathcal{M}_n^{-1})(f) \\ & = & V_{\beta,m}(W_{\alpha,n}(f)). \end{array} $$

■

iv. **Generalized Bessel-Struve transform.**

Definition 11. *The Generalized Bessel-Struve transform is defined on $L^1_{\alpha,n}(\mathbb{R})$ by*

$$ \forall \lambda \in \mathbb{R}, \quad \mathcal{F}^{\alpha,n}_{B,S}(f)(\lambda) = \int_{\mathbb{R}} f(x)\Psi_{-i\lambda,\alpha,n}(x)|x|^{2\alpha+1}dx. $$

Remark 7. • *It follows from (11), (23) and Definition 11 that $\mathcal{F}^{\alpha,n}_{B,S} = \mathcal{F}^{\alpha+2n}_{B,S} \circ \mathcal{M}_n^{-1}$, where $\mathcal{F}^{\alpha+2n}_{B,S}$ is the Bessel-Struve transform of order $\alpha + 2n$ given by (11).*

Proposition 12. *If $f \in L^1_{\alpha,n}(\mathbb{R})$ then*

(i) $\|\mathcal{F}^{\alpha,n}_{B,S}(f)\|_\infty \leq \|f\|_{1,\alpha,n}$.
(ii) $\mathcal{F}^{\alpha,n}_{B,S} = \mathcal{F} \circ W_{\alpha,n}$.

Proof. (i) By Remark 7 and Proposition 1, we have for all $f \in L^1_{\alpha,n}(\mathbb{R})$

$$ \begin{array}{rcl} \|\mathcal{F}^{\alpha,n}_{B,S}(f)\|_\infty & = & \|\mathcal{F}^{\alpha+2n}_{B,S}(\mathcal{M}_n^{-1}f)\|_\infty \\ & \leq & \|\mathcal{M}_n^{-1}f\|_{1,\alpha+2n} \\ & = & \|f\|_{1,\alpha,n}. \end{array} $$

(ii) From (27), Remark 7 and Proposition 2.(iii), we have for all $f \in L^1_{\alpha,n}(\mathbb{R})$

$$ \begin{array}{rcl} \mathcal{F}^{\alpha,n}_{B,S}(f) & = & \mathcal{F}^{\alpha+2n}_{B,S} \circ \mathcal{M}_n^{-1}(f) \\ & = & \mathcal{F} \circ W_{\alpha+2n}(\mathcal{M}_n^{-1}(f)) \\ & = & \mathcal{F} \circ W_{\alpha,n}(f). \end{array} $$

■

Proposition 13. *For all $f \in D_n(\mathbb{R})$, we have the following decomposition*

$$ \mathcal{F}^{\alpha,n}_{B,S}(f) = \mathcal{F}^{\beta,m}_{B,S} \circ \, {}^tS^{n,m}_{\alpha,\beta}(f). $$

Proof. It follows from (33), Remark 7 and Theorem 3.(iii) that

$$\begin{aligned}
\mathcal{F}_{B,S}^{\alpha,n}(f) &= \mathcal{F}_{B,S}^{\alpha+2n} \circ \mathcal{M}_n^{-1}(f) \\
&= \mathcal{F}_{B,S}^{\beta+2m} \circ {}^tS_{\alpha+2n,\beta+2m} \circ \mathcal{M}_n^{-1}(f) \\
&= \mathcal{F}_{B,S}^{\beta+2m}\mathcal{M}_m^{-1} \circ \mathcal{M}_m \circ {}^tS_{\alpha+2n,\beta+2m} \circ \mathcal{M}_n^{-1}(f) \\
&= \mathcal{F}_{B,S}^{\beta,m} \circ {}^tS_{\alpha,\beta}^{n,m}(f).
\end{aligned}$$

■

Theorem 9. *(Paley-Wiener) Let $a > 0$ and f a function in $\mathcal{D}_{a,n}(\mathbb{R})$ then $\mathcal{F}_{B,S}^{\alpha,n}$ can be extended to an analytic function on $\mathbb{C}$ that we denote again $\mathcal{F}_{B,S}^{\alpha,n}(f)$ verifying*

$$\forall k \in \mathbb{N}^*, \quad |\mathcal{F}_{B,S}^{\alpha,n}(f)(z)| \le Ce^{a|z|}.$$

Proof. The result follows directly from Remark 7, Lemma 1(i) and Theorem 2. ■

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