International Refereed Journal of Engineering and Science (IRJES)
ISSN (Online) 2319-183X, (Print) 2319-1821
Volume 14, Issue 5 (Sep. – Oct. 2025), PP. 49-59

Simulation of Quadratic Riccati Differential Equation Using Block Hybrid Method

Yusuf Skwame, Donald J. Zirra and Emmanuel James
Department of Mathematics, Adamawa State University, Mubi

Abstract
This research introduces a novel numerical method for solving quadratic Riccati differential equations using a block hybrid method based on power series. By applying a collocation strategy with power series, we develop a computational approach that offers effective approximation of solutions to this category of nonlinear differential equations. The proposed method is optimized to enhance both accuracy and stability, striking a balance between computational efficiency and precision. Through comprehensive stability analysis, we establish that the method is zero-stable and convergent, making it suitable for solving Riccati equations under various initial conditions and parameter settings. Numerical tests confirm the method's reliability by comparing its results with existing methods, demonstrating its potential as a dependable tool for solving nonlinear differential equations in fields such as applied mathematics and engineering.
Keywords: *Quadratic Riccati Differential Equations(QRDEs), Block Method, Multistage Variational Iteration Method (MVIM), Differential Transform Method (DTM), Non-Standard Finite Difference Method (NSFDM).*

Date of Submission: 08-09-2025 Date of acceptance: 19-09-2025

I. Introduction

Quadratic Riccati Differential Equations (QRDEs), initially explored by Count Jacopo Francesco Riccati, are a class of nonlinear ordinary differential equations widely applied in numerous disciplines, including engineering, finance, and network analysis (Riaz et al., 2021). These equations play a crucial role in areas that deal with stochastic processes, optimal control strategies, and diffusion phenomena. Their relevance extends to applications such as financial modeling, network design, and stochastic system realization (Baghchehjoughi et al., 2014; Nasr Al-Din, 2020b). Furthermore, QRDEs serve as effective models for representing complex physical systems like mass-spring arrangements, chemical kinetics, and pendulum motion. Their adaptability makes them essential tools in fields such as mathematical physics, control systems, and differential geometry for analyzing and interpreting real-world dynamics (Riaz et al., 2015; Vahidi and Didgar, 2018).

The Variational Iteration Method (VIM) and its enhanced form, the Multistage Variational Iteration Method (MVIM), have proven to be efficient techniques for solving Quadratic Riccati Differential Equations (QRDEs) by utilizing iterative correction processes to reach accurate solutions (Batiha, 2015). VIM operates based on an initial approximation and a linearization assumption, progressively improving the solution through a correction functional—making it effective for nonlinear differential equations such as QRDEs. MVIM builds upon this by introducing multiple stages of iteration, significantly enhancing both the accuracy and the rate of convergence (Hashim et al., 2016; Ghomanjani and Shateyi, 2020). This staged approach enables MVIM to handle the complexities of QRDEs more efficiently, providing refined results through repeated approximations. According to findings by Ghomanjani and Shateyi (2020), MVIM delivers superior precision compared to conventional VIM, highlighting its robustness in addressing a broad spectrum of nonlinear differential problems.

The Differential Transform Method (DTM) and the Non-Standard Finite Difference Method (NSFDM) offer alternative strategies for addressing Quadratic Riccati Differential Equations (QRDEs). DTM streamlines the problem-solving process by transforming differential equations into algebraic series, enabling a more manageable analysis of complex systems—an approach particularly beneficial in scientific and engineering contexts involving QRDEs (Sunday, 2017; Kamoh et al., 2020). In contrast, NSFDM introduces greater adaptability in numerical computation by employing unconventional finite difference schemes that utilize non-uniform grids and variable step sizes, thereby providing a closer representation of the actual behavior of the underlying differential equations. Riaz (2015) highlighted the effectiveness of NSFDM in QRDE applications by employing non-local approximations, which contribute to improved accuracy in numerical solutions.

The Adomian Decomposition Method (ADM) serves as an effective analytical technique for solving Quadratic Riccati Differential Equations (QRDEs) by breaking down the solution into a series of simpler subcomponents, often expressed in terms of a power series (Nasr Al-Din, 2020b; Agbata et al., 2021). Through its iterative decomposition strategy, ADM facilitates the resolution of complex nonlinear equations like QRDEs, making it widely adopted in disciplines that demand high levels of accuracy and stability. Research by Man et al. (2019) and Agbata et al. (2022) demonstrates ADM's capability in tackling such problems, while ongoing efforts, as noted by Wenjin and Yanni (2020), aim to enhance its precision and computational robustness. In response to existing limitations, this study proposes a novel approach built on power series functions, designed to boost computational efficiency, improve accuracy, and accelerate convergence in solving QRDEs (Ghomanjani and Shateyi, 2020).

II. Mathematical Formulation of the new Method

A novel numerical method were derived by estimating the solution expressed using power series as basis functions is given

$$y(t)=\sum_{j=0}^{r+s-1} a_j t^j \tag{2.1}$$

is consider, where $r\ and\ s$ are the numbers of collocation and interpolation points respectively.

Assume an approximate solution to equation (1.1) in the form of a power series of degree 10, by considering $r+s-1=10$ in equation (2.1), that is,

$$y(t)=\sum_{j=0}^{10} a_j t^j = a_0+a_1t+a_2t^2+a_3t^3+a_4t^4+a_5t^5+a_6t^6+a_7t^7+a_8t^8+a_9t^9+a_{10}t^{10} \tag{2.2}$$

Whose first derivative represented as

$$y'(t)=\sum_{j=0}^{10} ja_j t^{j-1} = a_1+2a_2t+3a_3t^2+4a_4t^3+5a_5t^4+6a_6t^5+7a_7t^6+8a_8t^7+9a_9t^8+10a_{10}t^9 \tag{2.3}$$

By inserting equations (2.2) and (2.3) into equation (1.1), we obtain

$$f(t)=a_0+a_1t+a_2t^2+a_3t^3+a_4t^4+a_5t^5+a_6t^6+a_7t^7+a_8t^8+a_9t^9+10a_{10}t^{10} \tag{2.4}$$

$$f(t,y)=a_1+2a_2t+3a_3t^2+4a_4t^3+5a_5t^4+6a_6t^5+7a_7t^6+8a_8t^7+9a_9t^8+10a_{10}t^9 \tag{2.5}$$

Now, interpolating (2.4) at point $t_{n+s}, s=1$ and collocating (2.5) at points $t_{n+r}=0\left(\frac{1}{9}\right)1$, this results in a system of nonlinear equations expressed in the form

$$TA=U \tag{2.6}$$

where

$$A=\begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & a_{10} \end{bmatrix}$$

$$U=\begin{bmatrix} y_{n+1} & f_n & f_{n+\frac{1}{9}} & f_{n+\frac{2}{9}} & f_{n+\frac{1}{3}} & f_{n+\frac{4}{9}} & f_{n+\frac{5}{9}} & f_{n+\frac{2}{3}} & f_{n+\frac{7}{9}} & f_{n+\frac{8}{9}} & f_{n+1} \end{bmatrix}^T$$

$$T=\begin{bmatrix}
1 & t_{n+1} & t_{n+1}^2 & t_{n+1}^3 & & t_{n+1}^5 & t_{n+1}^6 & t_{n+1}^7 & t_{n+1}^8 & t_{n+1}^9 & t_{n+1}^{10} \\
0 & 0 & 2t_n & 3t_n^2 & 4t_n^3 & 5t_n^4 & 6t_n^5 & 7t_n^6 & 8t_n^7 & 9t_n^8 & 10t_n^9 \\
0 & 0 & 2t_{n+\frac{1}{9}} & 3t^2_{n+\frac{1}{9}} & 4t^3_{n+\frac{1}{9}} & 5t^4_{n+\frac{1}{9}} & 6t^5_{n+\frac{1}{9}} & 7t^6_{n+\frac{1}{9}} & 8t^7_{n+\frac{1}{9}} & 9t^8_{n+\frac{1}{9}} & 10t^9_{n+\frac{1}{9}} \\
0 & 0 & 2t_{n+\frac{2}{9}} & 3t^2_{n+\frac{2}{9}} & 4t^3_{n+\frac{2}{9}} & 5t^4_{n+\frac{2}{9}} & 6t^5_{n+\frac{2}{9}} & 7t^6_{n+\frac{2}{9}} & 8t^7_{n+\frac{2}{9}} & 9t^8_{n+\frac{2}{9}} & 10t^9_{n+\frac{2}{9}} \\
0 & 0 & 2t_{n+\frac{1}{3}} & 3t^2_{n+\frac{1}{3}} & 4t^3_{n+\frac{1}{3}} & 5t^4_{n+\frac{1}{3}} & 6t^5_{n+\frac{1}{3}} & 7t^6_{n+\frac{1}{3}} & 8t^7_{n+\frac{1}{3}} & 9t^8_{n+\frac{1}{3}} & 10t^9_{n+\frac{1}{3}} \\
0 & 0 & 2t_{n+\frac{4}{9}} & 3t^2_{n+\frac{4}{9}} & 4t^3_{n+\frac{4}{9}} & 5t^4_{n+\frac{4}{9}} & 6t^5_{n+\frac{4}{9}} & 7t^6_{n+\frac{4}{9}} & 8t^7_{n+\frac{4}{9}} & 9t^8_{n+\frac{4}{9}} & 10t^9_{n+\frac{4}{9}} \\
0 & 0 & 2t_{n+\frac{5}{9}} & 3t^2_{n+\frac{5}{9}} & 4t^3_{n+\frac{5}{9}} & 5t^4_{n+\frac{5}{9}} & 6t^5_{n+\frac{5}{9}} & 7t^6_{n+\frac{5}{9}} & 8t^7_{n+\frac{5}{9}} & 9t^8_{n+\frac{5}{9}} & 10t^9_{n+\frac{5}{9}} \\
0 & 0 & 2t_{n+\frac{2}{3}} & 3t^2_{n+\frac{2}{3}} & 4t^3_{n+\frac{2}{3}} & 5t^4_{n+\frac{2}{3}} & 6t^5_{n+\frac{2}{3}} & 7t^6_{n+\frac{2}{3}} & 8t^7_{n+\frac{2}{3}} & 9t^8_{n+\frac{2}{3}} & 10t^9_{n+\frac{2}{3}} \\
0 & 0 & 2t_{n+\frac{7}{9}} & 3t^2_{n+\frac{7}{9}} & 4t^3_{n+\frac{7}{9}} & 5t^4_{n+\frac{7}{9}} & 6t^5_{n+\frac{7}{9}} & 7t^6_{n+\frac{7}{9}} & 8t^7_{n+\frac{7}{9}} & 9t^8_{n+\frac{7}{9}} & 10t^9_{n+\frac{7}{9}} \\
0 & 0 & 2t_{n+\frac{8}{9}} & 3t^2_{n+\frac{8}{9}} & 4t^3_{n+\frac{8}{9}} & 5t^4_{n+\frac{8}{9}} & 6t^5_{n+\frac{8}{9}} & 7t^6_{n+\frac{8}{9}} & 8t^7_{n+\frac{8}{9}} & 9t^8_{n+\frac{8}{9}} & 10t^9_{n+\frac{8}{9}} \\
0 & 0 & 2t_{n+1} & 3t^2_{n+1} & 4t^3_{n+1} & 5t^4_{n+1} & 6t^5_{n+1} & 7t^6_{n+1} & 8t^7_{n+1} & 9t^8_{n+1} & 10t^9_{n+1}
\end{bmatrix}$$

By applying the Gauss elimination method to solve the system of nonlinear equations $a_j's,\ j=0(1)10$ for the variable and then substituting the obtained results back into the power series basis function, we obtain a one-step block method with seven off-mesh points, presented in the following format

$$y(t)=a_1(t)y_{n+1}+h\sum_{j=0}^{1}\beta_j(t)f_{n+j},\ j=0\left(\frac{1}{9}\right)1 \tag{2.7}$$

where the coefficients of y_n *and* f_{n+j} are given as,

$$\alpha_1=1$$

$$\beta_0=t-\frac{2857}{89600}-\frac{7129}{560}t^2+\frac{19545}{224}t^3-\frac{40707}{112}t^4+\frac{12659}{128}t^5-\frac{2196477}{1280}t^6+\frac{885735}{448}t^7-\frac{5137263}{3584}t^8+\frac{531441}{896}t^9-\frac{4782969}{44800}t^{10}$$

$$\beta_{\frac{1}{9}}=-\frac{15741}{89600}+\frac{81}{2}t^2-\frac{124443}{280}t^3+\frac{10307331}{4480}t^4-\frac{5589243}{800}t^5+\frac{17073909}{1280}t^6-\frac{649539}{40}t^7+\frac{31355019}{2560}t^8-\frac{5845851}{1120}t^9-\frac{43046721}{44800}t^{10}$$

$$\beta_{\frac{2}{9}}=-\frac{27}{2240}-81t^2+\frac{158463}{140}t^3-\frac{15190173}{2240}t^4+\frac{18152829}{800}t^5-\frac{14843169}{320}t^6+\frac{33244587}{560}t^7-\frac{3720087}{80}t^8-\frac{22851963}{1120}t^9-\frac{43046721}{11200}t^{10}$$

$$\beta_{\frac{1}{3}}=-\frac{1209}{5600}+126t^2-\frac{18867}{10}t^3+\frac{1959363}{160}t^4-\frac{8776431}{200}t^5+\frac{30340251}{320}t^6-\frac{71035947}{560}t^7+\frac{16474671}{160}t^8-\frac{3720087}{80}t^9+\frac{14348907}{1600}t^{10}$$

$$\beta_{\frac{4}{9}}=\frac{2889}{44800}-\frac{567}{4}t^2+\frac{175473}{80}t^3-\frac{4752351}{320}t^4+\frac{89119521}{1600}t^5-\frac{80413803}{640}t^6+\frac{195629337}{1120}t^7-\frac{187598673}{1280}t^8-\frac{21789081}{320}t^9+\frac{43046721}{3200}t^{10}$$

$$\beta_{\frac{5}{9}}=-\frac{2889}{44800}+\frac{567}{5}t^2-\frac{7155}{4}t^3+\frac{795339}{64}t^4-\frac{3844017}{80}t^5+\frac{71674551}{640}t^6-\frac{18009945}{112}t^7+\frac{35606547}{256}t^8-\frac{531441}{8}t^9+\frac{43046721}{3200}t^{10}$$

$$\beta_{\frac{2}{3}}=\frac{1209}{5600}-63t^2+\frac{20127}{20}t^3-\frac{2276289}{320}t^4+\frac{22840173}{800}t^5-\frac{21482901}{320}t^6+\frac{55447011}{560}t^7-\frac{28166373}{320}t^8-\frac{6908733}{160}t^9+\frac{14348909}{1600}t^{10}$$

$$\beta_{\frac{7}{9}}=-\frac{27}{2240}+\frac{162}{7}t^2-\frac{26109}{70}t^3+\frac{2989629}{1120}t^4-\frac{2142531}{200}t^5+\frac{8347779}{320}t^6-\frac{22025277}{560}t^7+\frac{80247591}{2240}t^8-\frac{10097379}{560}t^9+\frac{43046721}{11200}t^{10}$$

$$\beta_{\frac{8}{9}}=-\frac{15741}{89600}-\frac{81}{16}t^2+\frac{91989}{1120}t^3-\frac{1328967}{2240}t^4+\frac{7712091}{3200}t^5-\frac{7626069}{1280}t^6+\frac{20490003}{2240}t^7-\frac{21789081}{2560}t^8-\frac{19663317}{4480}t^9+\frac{43046721}{44800}t^{10}$$

$$\beta_1=-\frac{2857}{89600}+\frac{1}{2}t^2-\frac{2283}{280}t^3+\frac{265779}{4480}t^4-\frac{19463}{800}t^5+\frac{779301}{1280}t^6-\frac{531441}{560}t^7+\frac{2302911}{2560}t^8-\frac{531441}{1120}t^9+\frac{4782969}{44800}t^{10}$$

and t is given by

$$x=\frac{t-t_n}{h} \tag{2.8}$$

Simplifying (2.7) at $t=\frac{1}{9}\left(\frac{1}{9}\right)1$, to gives the new method as

$$A^{(0)}\mathbf{Y}_m=E\mathbf{y}_n+hd\mathbf{f}(\mathbf{y}_n)+hb\mathbf{F}(\mathbf{Y}_m) \tag{2.9}$$

That is

$$\begin{bmatrix}1&0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0\\0&0&0&0&1&0&0&0&0\\0&0&0&0&0&1&0&0&0\\0&0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&0&1\end{bmatrix}\begin{bmatrix}y_{n+\frac{1}{9}}\\y_{n+\frac{2}{9}}\\y_{n+\frac{1}{3}}\\y_{n+\frac{4}{9}}\\y_{n+\frac{5}{9}}\\y_{n+\frac{2}{3}}\\y_{n+\frac{7}{9}}\\y_{n+\frac{8}{9}}\\y_{n+1}\end{bmatrix}=\begin{bmatrix}0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0&1\end{bmatrix}\begin{bmatrix}y_{n-\frac{1}{9}}\\y_{n-\frac{2}{9}}\\y_{n-\frac{1}{3}}\\y_{n-\frac{4}{9}}\\y_{n-\frac{5}{9}}\\y_{n-\frac{2}{3}}\\y_{n-\frac{7}{9}}\\y_{n-\frac{8}{9}}\\y_n\end{bmatrix}+h\begin{bmatrix}0&0&0&0&0&0&0&0&\frac{2857}{89600}\\0&0&0&0&0&0&0&0&\frac{3956}{127575}\\0&0&0&0&0&0&0&0&\frac{399}{12800}\\0&0&0&0&0&0&0&0&\frac{7}{225}\\0&0&0&0&0&0&0&0&\frac{81385}{2612736}\\0&0&0&0&0&0&0&0&\frac{7}{225}\\0&0&0&0&0&0&0&0&\frac{399}{12800}\\0&0&0&0&0&0&0&0&\frac{3956}{127575}\\0&0&0&0&0&0&0&0&\frac{2857}{89600}\end{bmatrix}\begin{bmatrix}f_{n-\frac{1}{9}}\\f_{n-\frac{2}{9}}\\f_{n-\frac{1}{3}}\\f_{n-\frac{4}{9}}\\f_{n-\frac{5}{9}}\\f_{n-\frac{2}{3}}\\f_{n-\frac{7}{9}}\\f_{n-\frac{8}{9}}\\f_n\end{bmatrix}$$

$$+h\begin{bmatrix} \frac{9449717}{65318400} & -\frac{1408913}{8164800} & \frac{200029}{816480} & -\frac{8641823}{32659200} & \frac{6755041}{32659200} & -\frac{462127}{4082400} & \frac{335983}{8164800} & -\frac{116687}{13063680} & \frac{8183}{9331200} \\ \frac{37829}{204120} & -\frac{34369}{510300} & \frac{45331}{255150} & -\frac{103987}{510300} & \frac{83291}{510300} & -\frac{9239}{102060} & \frac{8467}{255150} & -\frac{3697}{510300} & \frac{1}{1400} \\ \frac{16381}{89600} & -\frac{31}{1600} & \frac{13273}{50400} & -\frac{2123}{8960} & \frac{8201}{44800} & -\frac{5039}{50400} & \frac{407}{11200} & -\frac{101}{12800} & \frac{25}{32256} \\ \frac{3346}{18225} & -\frac{628}{25515} & \frac{40648}{127575} & -\frac{20924}{127575} & \frac{21076}{127575} & -\frac{11852}{127575} & \frac{872}{25515} & -\frac{953}{127575} & \frac{94}{127575} \\ \frac{478525}{2612736} & -\frac{7225}{1866240} & \frac{50425}{163296} & -\frac{131575}{1306368} & \frac{298505}{1306368} & -\frac{16775}{163296} & \frac{11975}{326592} & -\frac{20675}{2612736} & \frac{25}{32256} \\ \frac{257}{1400} & -\frac{17}{700} & \frac{199}{630} & -\frac{83}{700} & \frac{211}{700} & -\frac{299}{6300} & \frac{11}{350} & -\frac{1}{140} & \frac{1}{1400} \\ \frac{341383}{1866240} & -\frac{24647}{1166400} & \frac{178703}{583200} & -\frac{460649}{4665600} & \frac{1251607}{4665600} & \frac{4459}{116640} & \frac{92617}{1166400} & -\frac{90013}{9331200} & \frac{8183}{9331200} \\ \frac{23552}{127575} & -\frac{3712}{127575} & \frac{41984}{127575} & -\frac{3632}{25515} & \frac{41984}{127575} & -\frac{3712}{127575} & \frac{23552}{127575} & \frac{3956}{127575} & 0 \\ \frac{15741}{89600} & \frac{27}{2240} & \frac{1209}{5600} & \frac{2889}{44800} & \frac{2889}{44800} & \frac{1209}{5600} & \frac{27}{2240} & \frac{15741}{89600} & \frac{2857}{89600} \end{bmatrix} \begin{bmatrix} f_{n+\frac{1}{9}} \\ f_{n+\frac{2}{9}} \\ f_{n+\frac{1}{3}} \\ f_{n+\frac{4}{9}} \\ f_{n+\frac{5}{9}} \\ f_{n+\frac{2}{3}} \\ f_{n+\frac{7}{9}} \\ f_{n+\frac{8}{9}} \\ f_{n+1} \end{bmatrix}$$

III. Properties of the Novel Numerical Method

This section is dedicated to examining the fundamental properties of the novel numerical method, including its order, error constant, consistency, zero-stability, and region of absolute stability.

3.1 Order and Error Constant

Definition 3.1: Order and Error Constant

The linear operator associated with the new method is defined as follows

$$L\{y(t):h\} = \mathbf{A}^{(0)}\mathbf{Y}_m^{(i)} - \sum_{i-0}^{1} h^i e_i y_n^{(i)} - h^2\left(d_0 f(y_n) + b_0\mathbf{F}(\mathbf{Y}_m)\right) \qquad (3.1)$$

Assuming that $y(t)$ is sufficiently differentiable, we express the terms in (2.1) as a Taylor series expansion around the point t, yielding the following expression:

$$L\{y(t):h\} = c_0 y(t) + c_1 h y'(t) + c_2 h^2 y''(t) + \ldots + c_p h^p y^p(t) + c_{p+1} h^{p+1} y^{p+1}(t) + c_{p+1} h^{p+2} y^{p+2}(t) \quad (3.2)$$

By applying Definition 3.1 to the new method, the order of the method is determined to be

$p = [10 \quad 10 \quad 10 \quad 10 \quad 10 \quad 10 \quad 10 \quad 10 \quad 10]$. In other words, the method has a uniform order of 10.

The error constant is given by

$$C_{11} = \begin{bmatrix} -2.1624\times10^{-13} & -18097\times10^{-13} & -1.9236\times10^{-13} & -1.8632\times10^{-13} & -1.9124\times10^{-13} \\ -1.8520\times10^{-13} & -1.9658\times10^{-13} & -1.6132\times10^{-13} & -3.7756\times10^{-13} & \end{bmatrix}^T.$$

3.2 Consistency

As stated by Sunday (2017), the new method is consistent because it has a uniform order of 10.

3.3 Zero Stability

Definition 3.2

A new method is said to be zero-stable if the roots $z_s,\ s = 1, 2, \ldots, n$ of the first characteristic polynomial $\overline{\rho}(z)$, defined by

$$\overline{\rho}(z) = \det\left[zA^{(0)} - E\right] \qquad (3.2)$$

Satisfies $|z_s| \le 1$ and every root with $|z_s| = 1$ has multiplicity not exceeding the order of the differential equation as $h \to 0$. Moreover, as $h \to 0$, $\rho(z) = z^{r-\mu}(z-1)^{\mu}$, where μ is the order of the differential equation, r is the order of the matrices $A^{(0)}$ *and* E. The main role of zero stability is to control the error propagation during the integration process (Sunday, 2017). By applying Definition 3.2 to the new method, the first characteristic polynomial is given by

$$\rho(z)=\left| z\begin{bmatrix}1&0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0\\0&0&0&0&1&0&0&0&0\\0&0&0&0&0&1&0&0&0\\0&0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&0&1\end{bmatrix}-\begin{bmatrix}0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0&1\end{bmatrix}\right|=\begin{bmatrix}z&0&0&0&0&0&0&0&-1\\0&z&0&0&0&0&0&0&-1\\0&0&z&0&0&0&0&0&-1\\0&0&0&z&0&0&0&0&-1\\0&0&0&0&z&0&0&0&-1\\0&0&0&0&0&z&0&0&-1\\0&0&0&0&0&0&z&0&-1\\0&0&0&0&0&0&0&z&-1\\0&0&0&0&0&0&0&0&z-1\end{bmatrix}$$

$$=z^8(z-1)=0$$

Thus, solving for z in

$$z^8(z-1)=0 \tag{3.3}$$

Gives $z_1=z_2=z_3=z_4=z_5=z_6=z_7=z_8=0$ and $z_9=1$. Therefore, the new method is zero-stable.

3.4 Convergence

Theorem 3.1:

For a new method to achieve convergence, it must meet the essential criteria of both consistency and zero-stability (Kamoh et al., 2020). According to Theorem 3.1, the new method is considered convergent as it satisfies both of these condition.

3.5 Region of Absolute Stability

Definition 3.3:

Region of absolute stability is a region in the complex z plane, where $z=\lambda h$. It is defined as those values of z such that the numerical solutions of $y'=-\lambda y$ satisfy $y_j \to 0 \; as \; j \to \infty$ for any initial condition.

To determine the regions of absolute stability for the computational method, a technique was used that eliminates the need for computing polynomial roots or solving simultaneous inequalities. This approach is known as the Boundary Locus Method (Kamoh et al., 2020). By applying Definition 3.3 to the Boundary Locus Method, we obtain the stability polynomial for the new method as:

$$\bar{h}(w)=\left(-\frac{1}{3874204890}w^9+\frac{1}{3874204890}w^8\right)h^9+\left(-\frac{7129}{542388684600}w^8+\frac{7129}{542388684600}w^9\right)h^8+\left(-\frac{1303}{3214155168}w^9-\frac{1303}{3214155168}w^8\right)h^7 \tag{3.4}$$
$$+\left(-\frac{4523}{502211745}w^8+\frac{4523}{502211745}w^9\right)h^6+\left(-\frac{95}{629856}w^8-\frac{95}{629856}w^9\right)h^5+\left(-\frac{3013}{1574640}w^8+\frac{3013}{1574640}w^9\right)h^4+\left(-\frac{35}{1944}w^8-\frac{35}{1944}w^9\right)h^3$$
$$+\left(-\frac{29}{243}w^9+\frac{29}{243}w^{243}\right)h^2+\left(-\frac{1}{2}w^8-\frac{1}{2}w^9\right)h+w^9-w^8$$

The region of absolute stability for the new method is represented in Figure 3.1 as follows

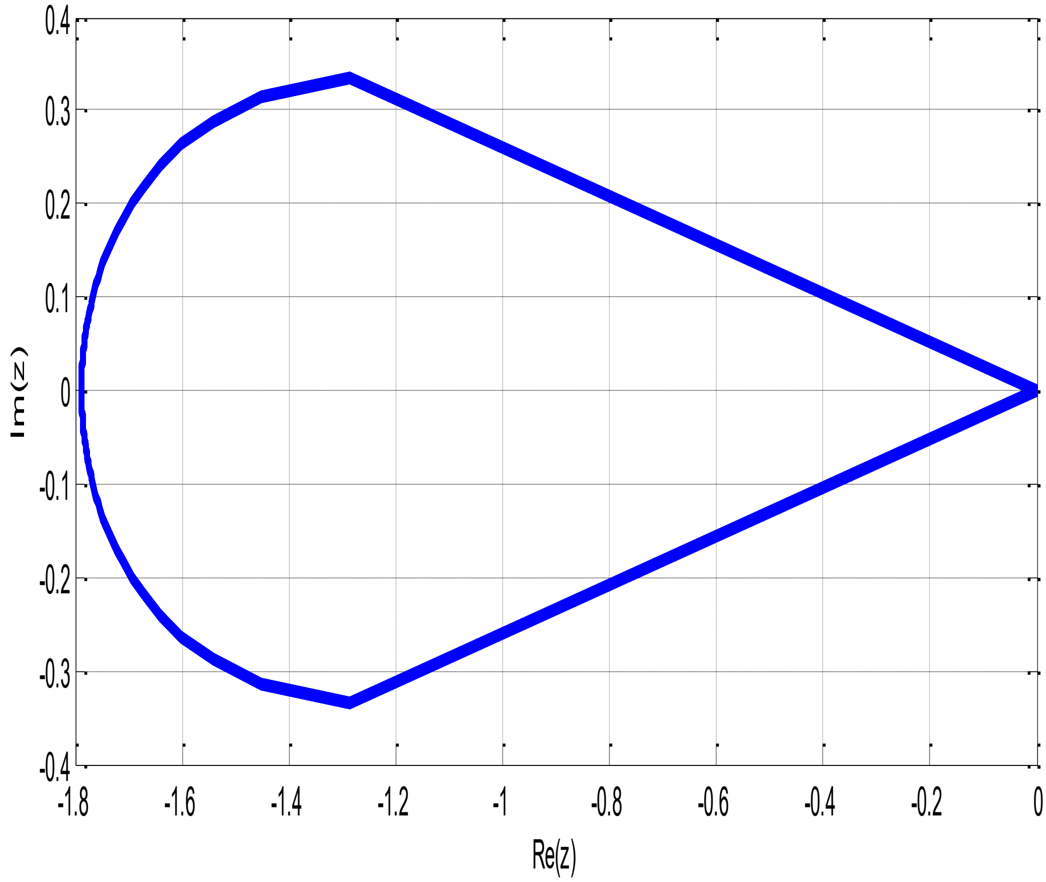

Figure 3.1: Illustrating the region of absolute stability

IV. Numerical Implementation of the Novel Numerical Method

We will apply the new method to several modeled Quadratic Riccati Differential Equations (QREs) of the form (1.1), as shown below. The results are then compared with those from Sunday (2017), File and Aga (2016), and Sunday and Philip (2018). The following acronyms will be used throughout the tables and figures.

Notations	Meaning
t	Point of Evaluation for time
ES	Exact Solution
CS	Computed Solution
ENM	Absolute Error in New Method
AES17	Absolute Error in Sunday, (2017)
EFA16	Absolute error in Fileand Aga, (2016)
ESP18	Absolute error in Sunday and Philip,(2018).

Problem 4.1

The nonlinear Quadratic Riccati Differential Equations considered by Sunday (2017) and File and Aga (2016) are of the form

$$y'(t) = -\frac{1}{1+t} + y - y^2(t), \quad y(0) = 1 \tag{4.1}$$

were selected, with their exact solutions given by

$$y(t) = \frac{1}{1+t} \tag{4.2}$$

Problem 4.2
This study considers the nonlinear Quadratic Riccati Differential Equations as presented in Sunday (2017) and Sunday and Philip (2018), given as

$$y'(t) = y^2(t) - 1, \quad y(0) = 0 \tag{4.3}$$

with exact solution as,

$$y(t) = -\tanh(t) \tag{4.4}$$

Problem 4.3
This study considers the nonlinear Quadratic Riccati Differential Equations as shown in Sunday, (2017), Sunday and Philip, (2018)

$$y'(t) = 1 - y^2(t), \quad y(0) = 0 \tag{4.5}$$

with exact solution is given by,

$$y(t) = \frac{e^{2t} - 1}{e^{2t} + 1} \tag{4.6}$$

Problem 4.4
This study considers the nonlinear Quadratic Riccati Differential Equations as shown in Sunday, (2017), Sunday and Philip, (2018)

$$y'(t) = 10 + 3y(t) - y^2(t), \quad y(0) = 0 \tag{4.7}$$

whose exact solution is given by,

$$y(t) = -2 + \frac{14e^{7t}}{5 + 2e^{7t}} \tag{4.8}$$

Table 4.1: Numerical Results for Problem 4.1 Compared with Sunday (2017) and File and Aga (2016)

t	ES	CS	ENM	AES17	AEFA16
0.100	0.90909090909090909091	0.90909090909090909090	1.0000e-20	2.2921e-12	3.8296e-07
0.200	0.83333333333333333333	0.83333333333333333338	5.0000e-20	3.1139e-12	3.8296e-07
0.300	0.76923076923076923077	0.76923076923076923079	2.000e-20	3.3764e-12	5.7951e-07
0.400	0.71428571428571428571	0.71428571428571428574	3.0000e-20	3.4242e-12	6.8133e-07
0.500	0.66666666666666666667	0.66666666666666666664	3.0000e-20	3.3944e-12	7.3394e-07
0.600	0.62500000000000000000	0.62500000000000000002	2.0000e-20	3.3436e-12	7.6091e-07
0.700	0.58823529411764705882	0.58823529411764705886	4.0000e-20	3.2949e-12	7.7483e-07
0.800	0.55555555555555555556	0.55555555555555555559	3.0000e-20	3.2574e-12	7.8257e-07
0.900	0.52631578947368421053	0.52631578947368421051	2.0000e-20	3.2344e-12	7.8799e-07
1.000	0.50000000000000000000	0.50000000000000000003	3.0000e-20	3.2265e-12	7.9326e-07

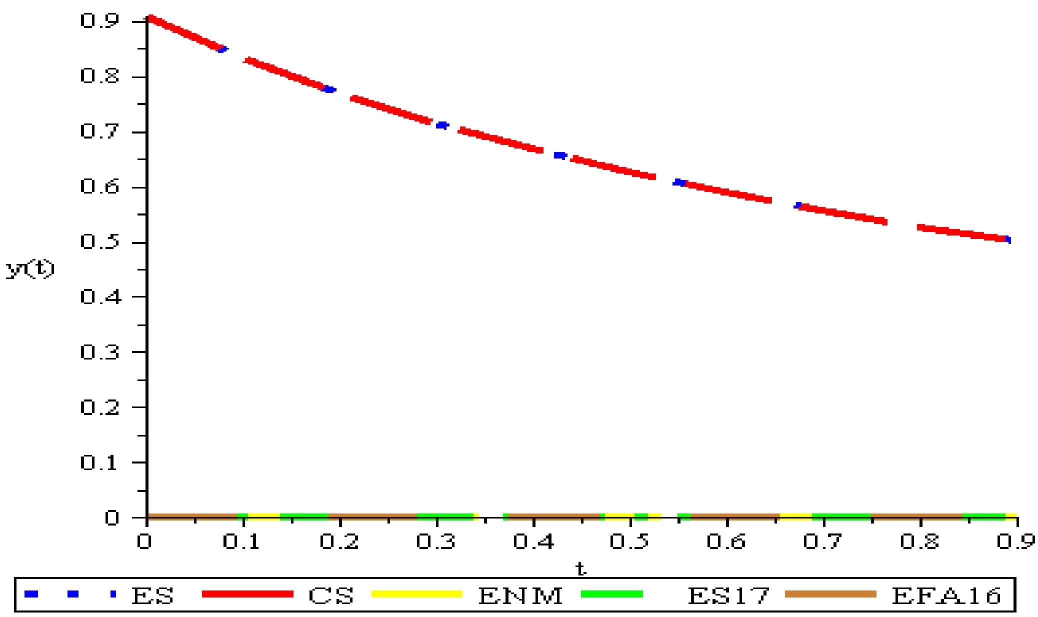

Figure 4.1: Graphical Representation of Table 4.1

Table 4.2: Numerical Results for Problem 4.2 Compared with Sunday (2017) and Sunday and Philip (2018)

t	ES	CS	ENM	AES17	AESP18
0.100	-0.09966799462495581712	-0.09966799462495581590	1.2200e-18	1.1477e-14	6.9389e-17
0.200	-0.19737532022490400074	-0.19737532022490399904	1.7000e-18	6.7141e-14	8.3267e-17
0.300	-0.29131261245159090582	-0.29131261245159090455	1.2700e-18	1.8341e-13	0.0000e00
0.400	-0.37994896225522488527	-0.37994896225522488492	3.5000e-19	3.3856e-13	2.2205e-16
0.500	-0.46211715726000975850	-0.46211715726000975896	4.6000e-19	4.8611e-13	1.1102e-16
0.600	-0.53704956699803528586	-0.53704956699803528666	8.0000e-19	5.7987e-13	2.2205e-16
0.700	-0.60436777711716349631	-0.60436777711716349702	7.1000e-19	5.9475e-13	2.2205e-16
0.800	-0.66403677026784896368	-0.66403677026784896409	4.1000e-19	5.3291e-13	5.5512e-16
0.900	-0.71629787019902442081	-0.71629787019902442090	9.0000e-20	4.1600e-13	4.4409e-16
1.000	-0.76159415595576488812	-0.76159415595576488810	2.0000e-20	2.7445e-13	2.2205e-16

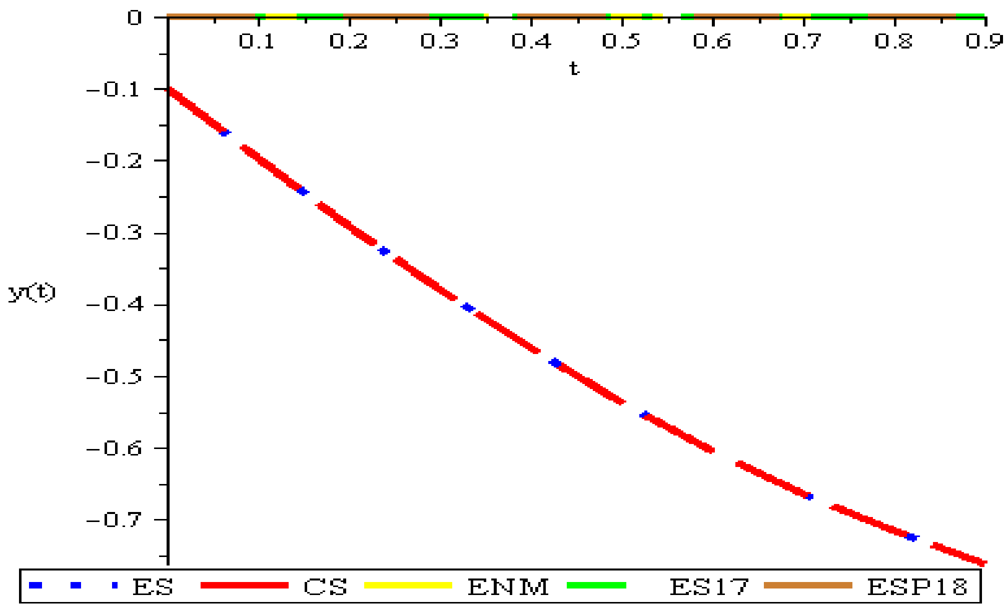

Figure 4.2: Graphical Representation of Table 4.2

Table 4.3: Numerical Results for Problem 4.3 Compared with Sunday, (2017), Sunday and Philip, (2018)

t	ES	CS	ENM	AES17	AESP18
0.100	0.09966799462495581711	0.09966799462495581590	1.2100e-18	1.1491e-14	9.7145e-17
0.200	0.19737532022490400073	0.19737532022490399904	1.6900e-18	6.7169e-14	8.3267e-17
0.300	0.29131261245159090582	0.29131261245159090455	1.2700e-18	1.8335e-13	0.0000e+00
0.400	0.37994896225522488527	0.37994896225522488492	3.5000e-19	3.3862e-13	2.2205e-16
0.500	0.46211715726000975851	0.46211715726000975896	4.5000e-19	4.8611e-13	2.2205e-16
0.600	0.53704956699803528586	0.53704956699803528666	8.0000e-19	5.7987e-13	1.1102e-16
0.700	0.60436777711716349631	0.60436777711716349702	7.1000e-19	5.9486e-13	3.3307e-16
0.800	0.66403677026784896369	0.66403677026784896409	4.0000e-19	5.3279e-13	5.5511e-16
0.900	0.71629787019902442081	0.71629787019902442090	9.0000e-20	4.1611e-13	5.5511e-16
1.000	0.76159415595576488812	0.76159415595576488810	2.0000e-20	2.7456e-13	3.3307e-16

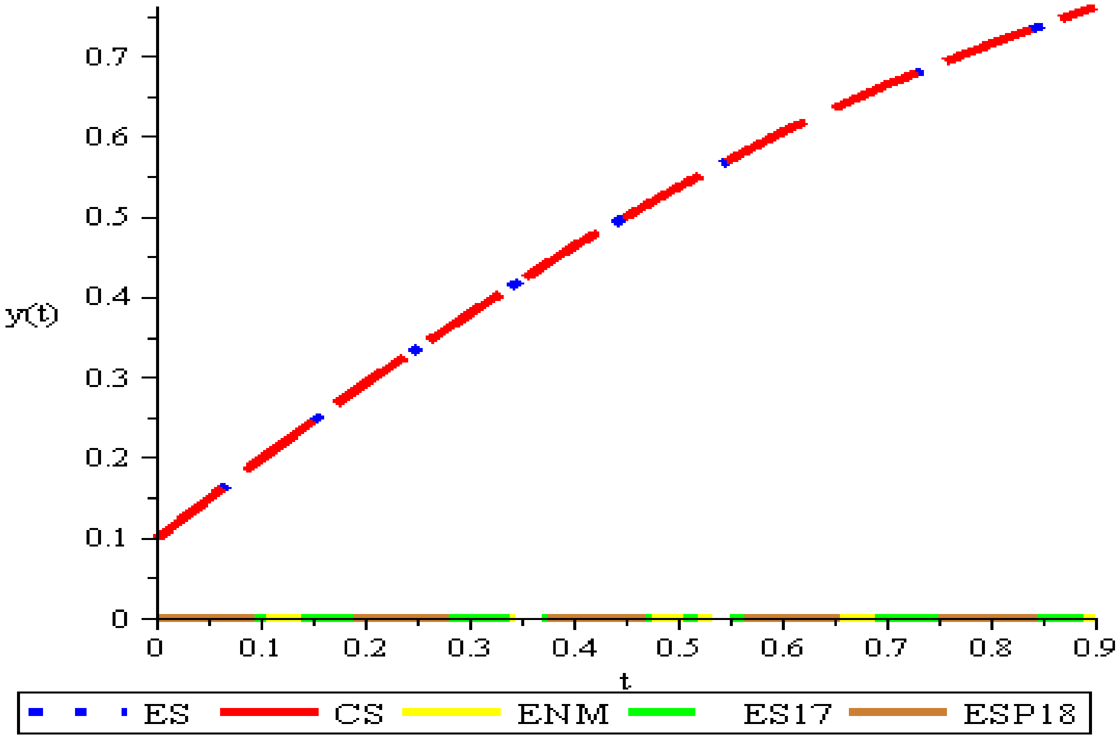

Figure 4.3: Graphical Representation of Table 4.3

Table 4.4: Numerical Results for Problem 4.4 Compared with Sunday (2017) and File and Aga (2016)

t	ES	CS	ENM	AES17	AESP18
0.100	1.12295995501998517310	1.12295995502210170750	2.11653e-12	2.82693e-09	1.46347e-09
0.200	2.33036366723934260660	2.33036366723771642140	1.62619e-12	5.89943e-09	2.99223e-09
0.300	3.35929859139218860340	3.35929859139349077430	1.30217e-12	6.83092e-08	3.49315e-08
0.400	4.07625619989394993700	4.07625619989411747890	1.67542e-13	1.49912e-07	7.66512e-08
0.500	4.50864023794231405830	4.50864023794228193720	3.21211e-14	1.83945e-07	9.40192e-08
0.600	4.74705986375186756050	4.74705986375192281450	5.52540e-14	1.65588e-07	8.46132e-08
0.700	4.87206646548954668230	4.87206646548958852320	4.18409e-14	1.24703e-07	6.37095e-08
0.800	4.93588015111826406050	4.93588015111828342610	1.93656e-14	8.43126e-08	4.30686e-08
0.900	4.96801151790818190370	4.96801151790819011510	8.21140e-15	5.32397e-08	2.71935e-08
1.000	4.98407836223863766150	4.98407836223864126670	3.60520e-15	3.21259e-08	1.64080e-08

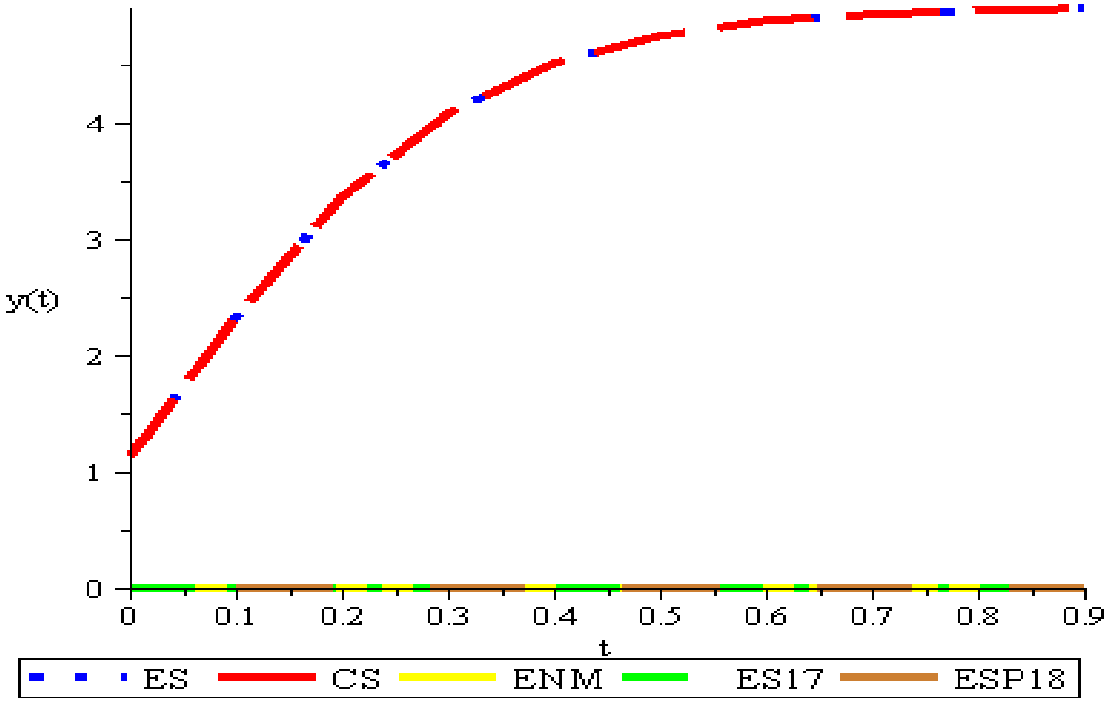

Figure 4.4: Graphical Representation of Table 4.4

V. Discussion of Results

The new method is developed by approximating the solution of a nonlinear Quadratic Riccati Differential Equation using a power series of degree 10, with interpolation and collocation applied at selected mesh and off-mesh points. By substituting the series and its derivative into the differential equation, a system of nonlinear equations is formed and solved using Gaussian elimination, leading to a one-step block method with seven off-mesh points. The method is analytically shown to possess a uniform order of 10, making it consistent and highly accurate. Further analysis confirms zero-stability, and by satisfying both consistency and zero-stability, the method is proven to be convergent. Its region of absolute stability is determined using the Boundary Locus Method, demonstrating the method's reliability for solving stiff and nonlinear problems.

The results from the comparison of the new method with previous methods for solving Quadratic Riccati Differential Equations (QREs) demonstrate its superior accuracy and efficiency. In Problem 4.1, the new method shows almost zero error at most time points, outperforming the solutions from Sunday (2017) and File and Aga (2016), which exhibit notably larger errors. This trend is clearly visualized in Figure 4.1, where the new method's solution closely follows the exact solution, while the previous methods show larger deviations.

Problem 4.2 further illustrates the effectiveness of the new method. The error (ENM) values for the new method remain significantly lower than those from Sunday (2017) and Sunday and Philip (2018), reinforcing its superior performance. At all time points, the new method maintains near-zero errors, while the other methods experience relatively larger errors. The graphical representation in Figure 4.2 further highlights these differences, showing the new method's stability and accuracy over time.

In Problem 4.3, the new method continues to outperform the previous methods. Its error remains close to zero, indicating its high precision in approximating the exact solution. The results for Sunday (2017) and Sunday and Philip (2018) show larger errors at various time points, with the new method providing a more reliable solution. The trend is again confirmed in Figure 4.3, where the new method closely follows the exact solution, emphasizing its accuracy and robustness.

Finally, Problem 4.4 consolidates the findings, with the new method demonstrating consistently lower errors than the previous methods from Sunday (2017) and File and Aga (2016). The graphical analysis in Figure 4.4 further supports these conclusions, visually confirming the new method's superior accuracy. Overall, the results across all four problems clearly show that the new method is more accurate and stable than the existing methods, making it a reliable tool for solving nonlinear QREs.

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