International Refereed Journal of Engineering and Science (IRJES)
ISSN (Online) 2319-183X, (Print) 2319-1821
Volume 5, Issue 8 (August 2016), PP.29-32

Structure of Multiplicatively Sub idempotent Semirings

M.Amala*[1], N.Sulochana[2] and T.Vasanthi[3]

[1,3]Dept. of Applied Mathematics, Yogi Vemana University, Kadapa, Andhra Pradesh, India
[2]Asst. Prof., K.S.R.M College of Engineering, Kadapa, Andhra Pradesh, India
E-mail: amalamaduri@gmail.com, sulochananagam@gmail.com and vasanthitm@gmail.com

Abstract: In this paper, we determine the structure of multiplicatively subidempotent semirings.

Keywords: Multiplicatively subidempotent semiring, zerosumfree semiring, regular semigroup, Band.

I. INTRODUCTION

The first formal definition of semiring was introduced in the year 1934 by Vandiver. However the developments of the theory in semirings have been taking place since 1950. A semiring is basic structure in Mathematics. The ring theory and semigroup theory influenced on the developments of the semiring theory and ordered semiring theory. Certainly the first mathematical structure we know–the natural number set N is a semiring. semirings play a significant role in different areas of mathematics as commutative, non-commutative ring theory, graph theory, combinatorics and the mathematical modeling of quantum physics and parallel computation systems. More details concerning such applications can be found in [7, 8, 9].

In a semiring S, an element ***a*** is Multiplicatively Subidempotent if $a + a^2 = a$. A semiring S is Multiplicatively Subidempotent if and only if each of its elements is Multiplicatively Subidempotent. Multiplicatively Subidempotent semiring plays an important role in modal logic.

Section one deals with introduction. Section two contains definitions. In last section we have given some results on multiplicatively subidempotent semiring and on its ordering.

II. PRELIMINARIES

A partially ordered semigroup (p. o. s. g) in which any two elements are comparable is totally ordered semigroup (t. o. s. g.) or fully ordered semigroup (f. o. s. g.). A triple (S, +, •) is a semiring if S is a non-empty set and "+, •" are binary operations on S satisfying that

(i) The additive reduct (S, +) and the multiplicative reduct (S, •) are semigroups.
(ii) $a (x + c) = ax + ac$ and $(x + c) a = xa + ca$, for all a, x, c in S.

A semiring (S, +, •) is denoted by S.

A semiring S is zerosumfree if $a + a = 0$, for all a in S. Here zero is an additive identity. A semiring S is zero-square if $a^2 = 0$, for all a in S. Here zero is multiplicative zero. A semigroup (S, •) is a band if $a^2 = a$ for all a in S. A semigroup (S, +) is a band if $a + a = a$ for all a in S. A semiring S is idempotent if $a + a = a$ and $a^2 = a$ for all a in S. A semiring S is totally ordered semiring (t.o.s.r) if there exists a partially order '≤ ' on S such that (i) (S, +) is a t. o. s. g. (ii) (S, •) is a t. o. s. g. It is usually denoted by (S, +, •, ≤). An element a in a partially ordered semigroup (S, •, ≤) is non-negative (non-positive) if $a^2 \geq a$ ($a^2 \leq a$). A partially ordered semigroup (S, •, ≤) is non-negatively ordered (non-positively ordered) if every element of S is non-negative (non-positive).

In a totally ordered semiring (S, +, •, ≤)

(i) (S, +, ≤) is positively totally ordered (p.t.o), if $a + x \geq a, x$ for all a, x in S

(ii) $(S, \bullet, \leq)$ is positively totally ordered (p.t.o), if $ax \geq a, x$ for all a, x in S.
(iii) $(S, +, \leq)$ is negatively totally ordered (n.t.o), if $a + x \leq a, x$ for all a, x in S
(iv) $(S, \bullet, \leq)$ is positively totally ordered (n.t.o), if $ax \leq a, x$ for all a, x in S.

III. STRUCTURE OF MULTIPLICATIVELY SUBIDEMPOTENT SEMIRINGS:

Proposition 3.1: Suppose 'a' in S is a multiplicatively subidempotent element.

(i) If 'a' satisfies the condition $x + a = x$ for all x in S. Then $x + a^n = x$ for all x in S and $n \geq 1$.
(ii) If S contains multiplicative identity '1' and 'a' is left cancellable with respect to $\bullet$, then $x + a^n x = x$ for all x in S and $n \geq 1$.
(iii) 'a' in (S, +) semigroup is regular and (S, +) is commutative, then $ax + a = a$.

Proof: **(i)** By hypothesis 'a' is multiplicatively subidempotent i..e $a + a^2 = a$
implies $x + a + a^2 = x + a$
Since $x + a = x$ we obtain $x + a^2 = x \rightarrow$ **(1)**
$\Rightarrow$ $x + a.a = x$ which implies $x + (a + a^2) a = x$
Using equation (1) in above we get $x + a^2 + a^3 = x$
The above equation reduces to $x + a^3 = x \rightarrow$ **(2)**
$\Rightarrow$ $x + (a + a^2) a^2 = x \Rightarrow x + a^3 + a^4 = x \Rightarrow x + a^4 = x \rightarrow$ **(3)**
Generalizing the above three equations we obtain $x + a^n = x$
Hence $x + a^n = x$ for all x in S and $n \geq 1$

(ii) Again let us consider $a + a^2 = a$
By hypothesis S contains multiplicative identity '1' then $a (1 + a) = a.1$
Since 'a' is left cancellable with respect to '$\bullet$' implies $1 + a =1$
$\Rightarrow x + ax = x \rightarrow$ **(1)**
Again let us take $a + a^2 = a \Rightarrow ax + a^2 x = ax$ this can also be written as $x + ax + a^2 x = x + ax$
Using equation (1) in above we get $x + a^2 x = x \rightarrow$ **(2)**
$\Rightarrow ax + a^3 x = ax \Rightarrow x + ax + a^3 x = x + ax$
$\Rightarrow x + a^3 x = x$ for all x in S $\rightarrow$ **(3)**
Continuing like this we obtain $x + a^n x = x$ for all x in S and $n \geq 1$
(iii) By hypothesis 'a' in S is multiplicatively subidempotent, $a + a^2 = a$
In (S, +) semigroup 'a' is regular then for an element 'a' there exists an element x in S such that $a + x + a = a$ implies $a (a + x + a) = a.a$
$\Rightarrow a^2 + ax + a^2 = a^2 \Rightarrow a + a^2 + ax + a^2 = a + a^2 \Rightarrow a + ax + a^2 = a$
Using (S, +) commutative in above we get $ax + a + a^2 = a \Rightarrow ax + a = a$
Thus $ax + a = a$ where 'x' depends on element 'a'

Theorem 3.2: Let S be a multiplicatively subidempotent semiring and (S, +) be left cancellative. Then $(S, \bullet)$ is a band if and only if (S, +) is a band.

Proof: Given that S is multiplicatively subidempotent semiring $a + a^2 = a$
Using $(S, \bullet)$ band in above we get $a + a = a$ for all a in S
Thus (S, +) is a band
To prove the converse part let us consider $a + a^2 = a$ for all a in S $\rightarrow$ **(1)**
Since $a + a = a$ for all a in S, then equation (1) reduces to $a + a^2 = a + a$
By applying left cancellation law we get $a^2 = a$ for all a in S
Thus $(S, \bullet)$ is a band

Theorem 3.3: Suppose S is a zerosumfree semiring with additive identity 0. Then S is multiplicatively subidempotent semiring if and only if S is a zero-square semiring.

Proof: By the definition of zerosumfree semiring a + a = 0 for all a in S

Also we have $a + a^2 = a$ for all a in S this implies $a + a + a^2 = a + a$

By using zerosumfree semiring in above we obtain $a^2 = 0$ for all a in S

Thus S is a zero-square semiring

To prove the converse part let us take $a^2 = 0$ implies $a + a^2 = a + 0$

i.e $a + a^2 = a$ for all a in S

Therefore S is multiplicatively subidempotent semiring

Theorem 3.4: Suppose S is a totally ordered multiplicatively subidempotent semiring and (S, +) is positively totally ordered (negatively totally ordered). If S contains multiplicative identity'1' and (S, •) is left cancellative, then 1 is the maximum (minimum) element.

Proof: By hypothesis we have $a + a^2 = a$ for all a in S which implies $a (1 + a) = a.1$

Using (S, •) left cancellation law in above we acquire 1 + a = 1

Also given that (S, +) is p.t.o implies $1 = 1 + a \geq a \Rightarrow 1 \geq a$, for $a \in S$

Thus 1 is the maximum element

In the same way if (S, +) is negatively totally ordered then 1 is the minimum element

Theorem 3.5: If S is a totally ordered multiplicatively subidempotent semiring, (S, •) is non-positively ordered (non-negatively ordered) then (S, +) is non-negatively ordered (non-positively ordered).

Proof: Given that S is multiplicatively subidempotent, $a + a^2 = a$

Since (S, •) is non-positively ordered $a^2 \leq a$ implies $a + a^2 \leq a + a$

The above equation can be reduced to $a \leq a + a \Rightarrow a + a \geq a$

Thus (S, +) is non-negatively ordered

Similarly if (S, •) is non-negatively ordered then (S, +) is non-positively ordered

REFERENCES

[1]. P.J.Allen, "A fundamental theorem of Homomorphism for Semirings", Proc. Amer. Math. Soc. 21 (1969), 412-416.

[2]. P.J.Allen, "An extension of the Hilbert Basis theorem to Semirings", Publ. Math. Debrecen, 22 (1975), 31-34.

[3]. S.Bourne, "On the homomorphism theorem for Semirings", Proc. Nat. Acad. Sci., U.S.A., Vol.38 (1952), 118- 119.

[4]. Ivan Chajda and M.Kotrle, "Boolean semirings", Czechoslovak Mathematical Journal, Vol.44 (1994), No.4, 763-767.

[5]. Stephen Dolan, "Fun with Semirings; A functional pearl on the abuse of linear algebra", Computer Laboratory, University of Cambridge 2013/7/17 PP 1-9.

[6]. Shamik Ghosh, "Another Note on the Least Lattice Congruence on Semirings", Soochow Journal of Mathematics" Vol.22, No.3 PP 357-362 July 1996.

[7]. Jonathan S.Golan, "The theory of Semirings with applications in Mathematics and Computer Science", Pitman Monographs and surveys in pure and applied mathematics, Vol – 54, Zongamann House, Burnt Mill, Harlow, Essex CM 20 JE. England: Longman Scientific and Technical, 1992.

[8]. Jonathan S.Golan, "Semirings and their Applications", Kluwer Academic Publishers (1999).

[9].Jonathan S.Golan, “Semirings and Affine Equations over Them: Theory and Applications”, Kluwer Academic Publishers (1999).

[10]. Y.Q.GUO and K.P.Shum, “The Semigroup Structure of Left Clifford Semirings”, Acta Mathematica Sinica, Oct., 2003, Vol.19, No.4, pp. 783–792.

[11].J.Hanumanthachari and T.Vasanthi, “On The Additive and Multiplicative Structure of Certain Classes of Ordered Semirings”, SEA. Bull. Math. Vol. 17.No.1 (1993), 3-10.

[12]. Peter Jipsen, “Categories of Algebraic Contexts Equivalent to Idempotent Semirings and Domain Semirings”, Springer-Verlag Berlin Heidelberg LNCS 7560, pp. 195–206, 2012.

[13]. M.Satyanarayana, “On the additive semigroup structure of semirings”, Semigroup Forum, Vol.23 (1981), 7-14.

[14]. M.Satyanarayana, “On the additive semigroup of ordered semirings”. Semigroup Forum, 31 (1985), 193-199.

[15]. K.P.Shum, “Topological semi primary semigroups”, Studia, Sci.Math., Hungar 15 (1980), No.1-3, 15-18.

[16]. K.P.Shum and C.S.Hoo, “Prime radical theorem on ordered Semigroup”, Semigroup Forum No.19 (1980), No.1, 87-94.

[17]. T.Vasanthi, “Semirings with IMP”, Southeast Asian Bulletin of Mathematics, 32(3) (2008), pp. 995 – 998.

[18]. W.B.Vasantha Kandasamy,“Smarandache Semirings, Semifields and Semivector Spaces”, Dept. of Mathematics, Indian Institute of Technology, Madras.

I.