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On A Class Of p −Valent Functions Involving Generalized Hypergeometric Function

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Abstract: *Invoking the Hadamard product (or convolution), a class of p-valent functions has been introduced. The coefficient bounds, extreme points, radii of close-to-convex, starlikeness and convexity are obtained for the same class of functions. Distortion theorem and fractional differ-integral operators are also obtained.*
Key words: *p-valent function; Analytic function; Hadamard product; Generalized hyper-geometric functions; Linear operator; Starlike function; Convex function; Fractional differential and integral operator.*

I. Introduction

Let $A(n,p)$ denote the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p}\, z^{n+p}\,, (p, n \in \mathrm{N} = \{1,2, \dots \dots\}), \tag{1.1}$$

which are analytic and p −valent in the unit disk $\mathbb{U} = \{z : |z| < 1\}$.
A function $f(z) \in A\,(n,p)$ is said to be p −valent starlike of order δ $(0 \le \delta < p)$ if and only if

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \delta\,, (\, z \in \mathbb{U};\ 0 \le \delta < p\,). \tag{1.2}$$

We denote by $S_n(p,\delta)$ the subclass of $A(n,p)$ consisting of functions which are p −valent starlike of order δ. Also a function $f(z) \in A(n,p)$ is said to be in the class $K_n(p,\delta)$ if and only if

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \delta\,, (z \in \mathbb{U}; 0 \le \delta < p). \tag{1.3}$$

A function $f(z) \in K_n(p,\delta)$ is called p -valent convex function of order δ (0≤ $\delta < p$), if

$$f(z) \in K_n(p,\delta) \Leftrightarrow zf'(z) \in S_n(p,\delta), \forall\, n \in N. \tag{1.4}$$

For a function $f(z)$ is given by (1.1) and $g(z) \in A(n,p)$ is given by

$$g(z) = z^p + \sum_{n=1}^{\infty} b_{n+p}\, z^{n+p}\,, \quad (\ p\,, n \in \mathrm{N}), \tag{1.5}$$

we define the Hadamard product of $f(z)$ and $g(z)$ as

$$(f * g)(z) = z^p + \sum_{n=1}^{\infty} a_{n+p}\, b_{n+p}\, z^{n+p}, \quad z \in \mathbb{U}. \tag{1.6}$$

Let the subclass $\mathrm{A}(n,p)$ is denoted by φ consisting of functions of the form

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{n+p}\, z^{n+p}\,, \quad (\,p\,, n \in \mathrm{N}; a_n \ge 0). \tag{1.7}$$

Invoking the Hadamard product, a linear operator $L_p^k(\,a,b,c)\,f(z)$ is defined as

$$L_p^k(\,a,b,c)\,f(z) = (z^p\,{}_2R_1\,(a,b;c;k;z)) * f(z)$$
$$= z^p - \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=1}^{\infty} \frac{(a)_n\,\Gamma(b+kn)}{\Gamma(c+kn)}\, \frac{a_{n+p}\,z^{n+p}}{(n)!} \tag{1.8}$$
$$= z^p - \sum_{n=1}^{\infty} a_{n+p}\, B(n) z^{n+p}\,, \tag{1.9}$$

where ${}_2R_1(a,b;c;z)$ is generalized hypergeomertric function defined by Virchenko, Kalla and Al-Zamel [6] as,

$$ {}_2R_1\,(a,b;c;k;z) = \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{(a)_n\ \Gamma(b+kn)}{\Gamma(c+kn)} \frac{z^n}{(n)!}, \qquad k \in R, k > 0, |z| < 1, \tag{1.10}$$

where $(a)_n$ is the pochhammer symbol defined as

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & (n=0) \\ a\,(a+1)\ldots\ldots(a+n-1); & (n \in N) \end{cases}$$

and

$$B(n) = \frac{\Gamma(c)}{\Gamma(b)} \frac{(a)_n\,\Gamma(b+kn)}{\Gamma(c+kn)\ (n)!}\ . \tag{1.11}$$

If we set $k = 1$ in (1.8), then the linear operator $L_p^k(\,a,b,c)\,f(z)$ reduces to the linear operator $L_p(\,a,b,c)\,f(z)$ defined (cf. [5]) as

$$L_p(\,a,b,c)\,f(z) = z^p - \sum_{n=1}^{\infty} \frac{(a)_n\,(b)_n\,a_{n+p}\,z^{n+p}}{(c)_n\,(n)!} = \left(z^p\,{}_2F_1\,(a,b;c;z)\right) * f(z). \tag{1.12}$$

In particular, if we set $b = 1$, then linear operator $L_p(\,a,b,c)\,f(z)$ reduces to Carlson-Shaffer operator $L(\,a,c)\,f(z)$ defined as

$$L_p(\,a,1,c)\,f(z) = L_p(\,a,c)\,f(z) = z^p - \sum_{n=1}^{\infty} \frac{(a)_n\ a_{n+p}\,z^{n+p}}{(c)_n}$$
$$= z^p\ \phi(a,c;z) * f(z), \tag{1.13}$$

where $\phi(a;c;z)$ is incomplete beta function defined as

$$\phi(a,c;z) = {}_2F_1(a,1;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n\ z^n}{(c)_n}.$$

For detail, one can see [1, 4].

A function $f(z) \in \varphi$, for $\alpha \geq 0$ and $0 \leq \beta < p$ is said to be in the class $A_p^k(\alpha,\beta)$ if it satisfies the following relation

$$\text{Re}\left\{\frac{z\left(L_p^k(\,a,b,c)\,f(z)\right)'}{L_p^k(\,a,b,c)\,f(z)}\right\} > \alpha\left|\frac{z\left(L_p^k(\,a,b,c)\,f(z)\right)'}{L_p^k(\,a,b,c)\,f(z)} - p\right| + \beta. \tag{1.14}$$

For detail, one can see [7].

II. Coefficient Bounds for Function $f(z)$ in $A_p^k(\alpha,\beta)$

Theorem 2.1. Let the function $f(z)$ defined by (1.7) be in the class $A_p^k(\alpha,\beta)$ if and only if

$$\sum_{n=1}^{\infty} \frac{[n(1+\alpha)+(p-\beta)]}{(p-\beta)} a_{n+p}\ B(n) < 1, \tag{2.1}$$

where $0 \leq \beta < p$, $\alpha \geq 0$ and $B(n)$ is defined by (1.11).

Proof. Let $f \in A_p^k(\alpha,\beta)$. By using Re $(w) > \alpha\,|w-p| + \beta$, if and only if, Re $(w(1+\alpha\,e^{i\gamma}) - p\alpha\,e^{i\gamma}) > \beta$ for real γ and letting

$$w = \frac{z\left(L_p^k(\,a,b,c)\,f(z)\right)'}{\left(L_p^k(\,a,b,c)\,f(z)\right)},$$

(1.14) reduces to

$$\left(\text{Re}\left(\frac{z\left(L_p^k(\,a,b,c)\,f(z)\right)'}{\left(L_p^k(\,a,b,c)\,f(z)\right)}\right)(1+\alpha\,e^{i\gamma}) - \text{p}\alpha e^{i\gamma}\right) > \beta,$$

or, equivalently,

$$\text{Re}\left(\frac{(p-\beta) - \sum_{n=1}^{\infty} B(n)[n+p-\beta]\,a_{n+p}\,z^n - \alpha e^{i\gamma} \sum_{n=1}^{\infty} n\,B(n) a_{n+p}\,z^n}{1 - \sum_{n=1}^{\infty} B(n) a_{n+p}\,z^n}\right) > 0. \tag{2.2}$$

The inequality (2.2) must hold for all z in $\mathbb{U}$. By letting $z \to 1^-$, we have

$$\mathrm{Re}\left(\frac{(p-\beta)-\sum_{n=1}^{\infty} B(n)\,[n+p-\beta]a_{n+p}-\alpha e^{i\gamma}\sum_{n=1}^{\infty} n\,B(n)a_{n+p}}{1-\sum_{n=1}^{\infty} B(n)a_{n+p}}\right) > 0,$$

by means of mean value theorem, we get

$$\left((p-\beta)-\sum_{n=1}^{\infty} a_{n+p}\ B(n)\,[n+p-\beta]-\alpha\sum_{n=1}^{\infty} n\,a_{n+p}\ B(n)\right) \geq 0.$$

Therefore,

$$\sum_{n=1}^{\infty}[n(1+\alpha)+(p-\beta)]\,a_{n+p}\ B(n) \leq (p-\beta). \tag{2.3}$$

Conversely, we assume that (2.1) hold. We are to show that (1.14) is satisfied and so $f \in A_p^k(\alpha,\beta)$. By using $\mathrm{Re}\,(w) > \alpha$ if and only if $|\,w-(p+\alpha)\,| < |\,w+(p-\alpha)\,|$, it is sufficient to show that

$$\left|\left(\frac{z\left(L_p^k(a,b,c)\,f(z)\right)'}{\left(L_p^k(a,b,c)\,f(z)\right)}\right)-\left(p+\alpha\left|\left(\frac{z\left(L_p^k(a,b,c)\,f(z)\right)'}{\left(L_p^k(a,b,c)\,f(z)\right)}\right)-p\right|+\beta\right)\right|$$
$$< \left|\left(\frac{z\left(L_p^k(a,b,c)\,f(z)\right)'}{\left(L_p^k(a,b,c)\,f(z)\right)}\right)+\left(p-\alpha\left|\left(\frac{z\left(L_p^k(a,b,c)\,f(z)\right)'}{\left(L_p^k(a,b,c)\,f(z)\right)}\right)-p\right|-\beta\right)\right|.$$

Let

$$e^{i\emptyset} = \frac{\left(L_p^k(a,b,c)\,f(z)\right)}{\left|\left(L_p^k(a,b,c)\,f(z)\right)\right|}$$

then,

$$\mathrm{E} = \left|\left(\frac{z\left(L_p^k(a,b,c)\,f(z)\right)'}{\left(L_p^k(a,b,c)\,f(z)\right)}\right)+\left(p-\alpha\left|\left(\frac{z\left(L_p^k(a,b,c)\,f(z)\right)'}{\left(L_p^k(a,b,c)\,f(z)\right)}\right)-p\right|-\beta\right)\right|$$

$$= \frac{\left|z\left(L_p^k(a,b,c)\,f(z)\right)'+(p-\beta)\left(L_p^k(a,b,c)\,f(z)\right)-\alpha e^{i\emptyset}\left|z\left(L_p^k(a,b,c)\,f(z)\right)'-p\,(L_p^k(a,b,c)\,f(z))\right|\right|}{\left|\left(L_p^k(a,b,c)\,f(z)\right)\right|}$$

$$> \frac{|z^p|\left\{2p-\beta-\sum_{n=1}^{\infty}[2p-\beta+n(1-\alpha)]B(n)\,a_{n+p}\ |z|^n\right\}}{\left|\left(L_p^k(a,b,c)\,f(z)\right)\right|}. \tag{2.4}$$

and

$$\mathrm{F} = \left|\left(\frac{z\left(L_p^k(a,b,c)\,f(z)\right)'}{\left(L_p^k(a,b,c)\,f(z)\right)}\right)-\left(p+\alpha\left|\left(\frac{z\left(L_p^k(a,b,c)\,f(z)\right)'}{\left(L_p^k(a,b,c)\,f(z)\right)}\right)-p\right|+\beta\right)\right|$$

$$= \frac{\left|z\left(L_p^k(a,b,c)\,f(z)\right)'-(p+\beta)\left(L_p^k(a,b,c)\,f(z)\right)-\alpha e^{i\emptyset}\left|z\left(L_p^k(a,b,c)\,f(z)\right)'-p\left(L_p^k(a,b,c)\,f(z)\right)\right|\right|}{\left|\left(L_p^k(a,b,c)\,f(z)\right)\right|}$$

$$> \frac{|z^p|\left\{\beta+\sum_{n=1}^{\infty}[n(1+\alpha)-\beta]\,a_{n+p}\ B(n)|z|^n\right\}}{\left|\left(L_p^k(a,b,c)\,f(z)\right)\right|} \tag{2.5}$$

if (2.1) holds, then it is easy to show that E – F > 0. So the proof is completed.

III. Extreme Points for the class $A_p^k(\alpha,\beta)$

Theorem 3.1. Let $f_1(z) = z^p$ (3.1) and $f_n(z) = z^p - \frac{p-\beta}{[n(1+\alpha)+(p-\beta)]B(n)} z^{n+p}$, (3.2)

where $n \in N$, then $f(z) \in A_p^k(\alpha,\beta)$, if and only if, $f(z)$ can be expressed as

$$f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z), \tag{3.3}$$

where

$$\lambda_1 + \sum_{n=1}^{\infty} \lambda_n = 1, \quad (\lambda_1 \geq 0;\ \lambda_n \geq 0;\ n \in N) \tag{3.4}$$

and $B(n)$ is defined by (1.11).

Proof. Invoking (3.1), (3.2) and (3.3), we get

$$f(z) = \lambda_1 z^p + \sum_{n=1}^{\infty} \lambda_n \left\{ z^p - \frac{p-\beta}{[n(1+\alpha)+(p-\beta)]B(n)} z^{n+p} \right\}$$

$$f(z) = z^p - \sum_{n=1}^{\infty} t_{n+p} z^{n+p}, \tag{3.5}$$

where

$$t_{n+p} = \frac{p-\beta}{[n(1+\alpha)+(p-\beta)]B(n)} \lambda_n$$

By using (3.4), we get

$$\sum_{n=1}^{\infty} \frac{[n(1+\alpha)+(p-\beta)]B(n)}{p-\beta} t_{n+p} = 1 - \lambda_1 < 1, \tag{3.6}$$

hence $f \in A_p^k(\alpha,\beta)$.

Conversely, if $f \in A_p^k(\alpha,\beta)$, then by using theorem 2.1, we get

$$a_{n+p} \leq \frac{p-\beta}{[n(1+\alpha)+(p-\beta)]B(n)}, (n \in N), \tag{3.7}$$

then

$$\frac{[n(1+\alpha)+(p-\beta)]B(n)}{p-\beta} \lambda_n a_{n+p} < 1, (n \in N) \tag{3.8}$$

and $\lambda_1 = 1 - \sum_{n=1}^{\infty} \lambda_n$.

i.e.

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{n+p} z^{n+p}$$

$$= z^p - \sum_{n=1}^{\infty} \frac{p-\beta}{[n(1+\alpha)+(p-\beta)]B(n)} z^{n+p}$$

Using (3.2) and (3.4), we get

$$f(z) = z^p - \sum_{n=1}^{\infty} \lambda_n \left(z^p - f_n(z) \right)$$

$$f(z) = z^p \left(1 - \sum_{n=1}^{\infty} \lambda_n \right) + \sum_{n=1}^{\infty} \lambda_n f_n(z).$$

Theorem is completely proved.

IV. Distortion Bounds for the $L_p^k(a,b,c)\, f(z)$

Theorem 4.1. Let the function $f(z)$ defined by (1.7) be in the class $A_p^k(\alpha,\beta)$. Then

$$z^p - |z|^{1+p} \frac{p-\beta}{[(1+\alpha)+(p-\beta)]} \leq \left| \left(L_p^k(a,b,c)\, f(z) \right) \right| \leq |z|^p + |z|^{1+p} \frac{p-\beta}{[(1+\alpha)+(p-\beta)]} \tag{4.1}$$

Proof. Let $f \in A_p^k(\alpha,\beta)$. Then by using theorem 2.1, we have

$$\sum_{n=1}^{\infty} a_{n+p} B(n) \leq \frac{p-\beta}{[(1+\alpha)+(p-\beta)]}, (n \geq 1) \tag{4.2}$$

now by using (1.9), we get

$$\left|\left(L_p^k(a,b,c)\,f(z)\right)\right| \le |z|^p + \sum_{n=1}^{\infty} a_{n+p}\,B(n)|z|^{n+p}$$
$$\le |z|^p + \frac{p-\beta}{[(1+\alpha)+(p-\beta)]}|z|^{1+p}$$

and

$$\left|\left(L_p^k(a,b,c)\,f(z)\right)\right| \ge |z|^p - \sum_{n=1}^{\infty} a_{n+p}\,B(n)|z|^{n+p}$$
$$\ge |z|^p - \frac{p-\beta}{[(1+\alpha)+(p-\beta)]}|z|^{1+p}.$$

Theorem is completely proved.

V. Radius of close –to- convexity, starlikeness and convexity

Theorem 5.1. Let the function $f(z)$ defined by (1.7) be in the class $A_p^k(\alpha,\beta)$. Then $f(z)$ is $p-$ valently close to convex of order $\delta(0\le \delta < p)$ in $|z| < r_1(\alpha,\beta)$, where

$$r_1(\alpha,\beta) = \inf_n\left[\frac{(p-\delta)[n(1+\alpha)+(p-\beta)]B(n)}{(n+p)(p-\beta)}\right]^{1/n}, n\ge 1 \qquad (5.1)$$

and $B(n)$ is defined by (1.11).

Proof. It is sufficient to prove that

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| \le p-\delta \text{ , for } |z| < r_1(\alpha,\beta),$$

we have

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| \le \sum_{n=1}^{\infty}(n+p)\,a_{n+p}\,|z|^n.$$

Thus $\left|\frac{f'(z)}{z^{p-1}} - p\right| \le p-\delta$ if

$$\sum_{n=1}^{\infty}\frac{(n+p)a_{n+p}}{(p-\delta)}|z|^n \le 1,$$

but by theorem 2.1, above inequality hold true if

$$\frac{(n+p)}{(p-\delta)}|z|^n \le \frac{[n(1+\alpha)+(p-\beta)]B(n)}{(p-\beta)},$$

$$|z| \le \left\{\frac{(p-\delta)[n(1+\alpha)+(p-\beta)]B(n)}{(n+p)(p-\beta)}\right\}^{1/n}, n\ge 1$$

or

$$r_1(\alpha,\beta) = \inf_n\left[\frac{(p-\delta)[n(1+\alpha)+(p-\beta)]B(n)}{(n+p)(p-\beta)}\right]^{1/n}, n\ge 1.$$

The Theorem is completely proved.

Theorem 5.2. Let the function $f(z)$ defined by (1.7) be in the class $A_p^k(\alpha,\beta)$. Then $f(z)$ is $p-$ valently starlike of order $\delta(0\le \delta < p)$ in $|z| < r_2(\alpha,\beta)$, where

$$r_2(\alpha,\beta) = \inf_n\left[\frac{(p-\delta)[n(1+\alpha)+(p-\beta)]B(n)}{(n+p-\delta)(p-\beta)}\right]^{1/n}, n\ge 1 \qquad (5.2)$$

and $B(n)$ is defined by (1.11).

Proof. It is sufficient to prove that

$$\left|\frac{zf'(z)}{f(z)} - p\right| \le p-\delta \text{ , } (0\le\delta< p)$$

we have

$$\left|\frac{zf'(z)}{f(z)} - p\right| \le \frac{\sum_{n=1}^{\infty} n\,a_{n+p}\,|z|^n}{1-\sum_{n=1}^{\infty} a_{n+p}\,|z|^n}.$$

Thus $\left|\frac{zf'(z)}{f(z)} - p\right| \le p-\delta$ if

$$\sum_{n=1}^{\infty}\frac{(n+p-\ \delta)a_{n+p}}{(p-\delta)}|z|^n\le 1,$$

but by theorem 2.1, above inequality hold true if

$$|z|\ \le\ \left\{\frac{(p-\delta\ )[n(1+\alpha)+(p-\beta)]B(n)}{(n+p-\ \delta)(p-\beta)}\right\}^{1/n},n\ge 1$$

or

$$r_2(\alpha,\beta)\ =\ \inf_n\left[\frac{(p-\delta)[n(1+\alpha)+(p-\beta)]B(n)}{(n+p-\delta)(p-\beta)}\right]^{1/n},n\ge 1.$$

The Theorem is completely proved.

Corollary 1. Let the function $f(z)$ defined by (1.7) be in the class $A_p^k(\alpha,\beta)$. Then $f(z)$ is $p-$ valently convex of order $\delta(0\le\ \delta<p)$ in $|z|<r_3(\alpha,\beta)$, where

$$r_3(\alpha,\beta)=\inf_n\left[\frac{p\ (p-\delta)[n(1+\alpha)+(p-\beta)]B(n)}{(n+p)(n+p-\delta)(p-\beta)}\right]^{1/n},n\ge 1 \qquad (5.3)$$

and $B(n)$ is defined by (1.11).

VI. Fractional integral operator

The following definition given by Srivastava and Owa [2] are required to prove the results in this section:

Definition 1. The fractional derivative of order α for a function $f(z)$ is defined by

$$D_z^{\alpha}f(z)=\frac{1}{\Gamma(1-\alpha)}\ \frac{d}{dz}\int_0^z\frac{f(t)}{(z-t)^{\alpha}}dt\ ;0\le\ \alpha<1, \qquad (6.1)$$

where the function $f(z)$ is analytic in simply-connected region of the z -plane containing the origin and multiplicity of $(z-t)^{-\alpha}$ is removed by requiring $\log(z-t)$ to be real when $(z-t)>0$.

Definition 2. The fractional integral of order α is defined , for a analytic function $f(z)$ by

$$D_z^{-\alpha}f(z)=\frac{1}{\Gamma(\alpha)}\ \int_0^z\frac{f(t)}{(z-t)^{1-\alpha}}dt\ ;\alpha>0\ , \qquad (6.2)$$

where the analytic function $f(z)$ is in a simply-connected region of the z -plane containing the origin and multiplicity of $(z-t)^{\alpha-1}$ is removed by requiring $\log(z-t)$ to be real when $(z-t)>0$.

Definition 3. By the hypothesis of definition 1, the fractional derivative of function $f(z)$ for order $k+\alpha$ is defined by

$$D_z^{k+\alpha}f(z)=\frac{d^k}{dz^k}D_z^{\alpha}f(z),(0\le\alpha<1;k\in\ N_0). \qquad (6.3)$$

We also need the following definition of fractional integral operator given by Srivastava, Saigo and Owa [3] defined as

Definition 4. The fractional operator $I_{0,z}^{\alpha,\eta,\delta}$ for real numbers >0 , $\eta\ and\ \delta$ is defined by

$$I_{0,z}^{\alpha,\eta,\delta}\ f(z)=\frac{z^{-\alpha-\eta}}{\Gamma(\alpha)}\int_0^z(z-t)\,{}_2F_1\left(\alpha+\eta,-\delta;\ \alpha\ ;1-\frac{t}{z}\right)f(t)dt, \qquad (6.4)$$

where the function $f(z)$ is analytic in simply-connected region of the z -plane containing the origin with order $f(z)=\mathrm{O}(|z|^{\varepsilon})$, $z\to 0$ and $\varepsilon>\max(0,\eta-\delta)-1$,

$${}_2F_1\,(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n\,(b)_n}{(c)_n}\frac{z^n}{n!},$$

where $(a)_n$ is the Pochhammer symbol and multiplicity of $(z-t)^{\alpha-1}$ is removed by requiring $\log(z-t)$ to be real when $(z-t)>0$.

Lemma 1. If $\alpha>0$ and $n\ >\eta-\delta-1$, then

$$I_{0,z}^{\alpha,\eta,\delta}\,z^n=\frac{\Gamma(n+1)\,\Gamma(n-\eta+\delta+1)}{\Gamma(n-\eta+1)\ \ \Gamma(n+\alpha+\delta+1)}z^{n-\eta}. \qquad (6.5)$$

Theorem 6.1. Let $f(z)$ defined by (1.7) be in the class $A_p^k(\alpha,\beta)$ for $\alpha>0,\eta<2$, then

$$\left|I_{0,z}^{\alpha,\eta,\delta}f(z)\right|\ge \frac{\Gamma(p+1)\,\Gamma(p-\eta+\delta+1)}{\Gamma(p-\eta+1)\ \ \Gamma(p+\alpha+\delta+1)}\,|z|^{p-\eta}\ \left(1-\frac{(p+1)\,(p-\beta)(p+1-\eta+\delta)}{(p+1-\eta)(p+1+\alpha+\delta\ )(p+1+\alpha-\beta)\ a}\frac{\Gamma(c+k)}{\Gamma(b+k)}\frac{\Gamma(b)}{\Gamma(c)}|z|\right) \qquad (6.6)$$

and

$$\left|I_{0,z}^{\alpha,\eta,\delta}f(z)\right|\le \frac{\Gamma(p+1)\,\Gamma(p-\eta+\delta+1)}{\Gamma(p-\eta+1)\ \ \Gamma(p+\alpha+\delta+1)}|z|^{p-\eta}\left(1+\frac{(p+1)\,(p-\beta)(p+1-\eta+\delta)}{(p+1-\eta)(p+1+\alpha+\delta\ )(p+1+\alpha-\beta)}\frac{\Gamma(c+k)}{a\Gamma(b+k)}\frac{\Gamma(b)}{\Gamma(c)}|z|\right). \qquad (6.7)$$

Proof. By using lemma 1, we get

$$I_{0,z}^{\alpha,\eta,\delta}f(z)=\frac{\Gamma(p+1)\,\Gamma(p-\eta+\delta+1)}{\Gamma(p-\eta+1)\ \ \Gamma(p+\alpha+\delta+1)}z^{p-\eta}\ -\sum_{n=1}^{\infty}\ \frac{\Gamma(n+p+1)\,\Gamma(n+p-\eta+\delta+1)}{\Gamma(n+p-\eta+1)\ \ \Gamma(n+p+\alpha+\delta+1)}\,a_{n+p}\ z^{n+p-\eta} \qquad (6.8)$$

$$\frac{\Gamma(p-\eta+1)\ \Gamma(p+\alpha+\delta+1)}{\Gamma(p+1)\,\Gamma(p-\eta+\delta+1)} z^{\eta}\ I_{0,z}^{\alpha,\eta,\delta} f(z)$$
$$= z^p - \sum_{n=1}^{\infty} \frac{\Gamma(n+p+1)\,\Gamma(n+p-\eta+\delta+1)}{\Gamma(n+p-\eta+1)\ \Gamma(n+p+\alpha+\delta+1)} \frac{\Gamma(p-\eta+1)\ \Gamma(p+\alpha+\delta+1)}{\Gamma(p+1)\,\Gamma(p-\eta+\delta+1)} a_{n+p}\, z^{n+p},$$

let

$$H(z) = \frac{\Gamma(p-\eta+1)\ \Gamma(p+\alpha+\delta+1)}{\Gamma(p+1)\,\Gamma(p-\eta+\delta+1)} z^{\eta}\ I_{0,z}^{\alpha,\eta,\delta} f(z) = z^p - \sum_{n=1}^{\infty} h(n)\, a_{n+p}\ z^{n+p}\,, \quad (6.9)$$

where

$$h(n) = \frac{\Gamma(n+p+1)\,\Gamma(n+p-\eta+\delta+1)}{\Gamma(n+p-\eta+1)\ \Gamma(n+p+\alpha+\delta+1)} \frac{\Gamma(p-\eta+1)\ \Gamma(p+\alpha+\delta+1)}{\Gamma(p+1)\,\Gamma(p-\eta+\delta+1)}$$
$$= \frac{(p+1)_n\ (p+1-\eta+\delta)_n}{(p+1-\eta)_n\,(p+1+\alpha+\delta)_n}, (n\ \geq 1), \quad (6.10)$$

we can see that h(n) is decreasing for $n \geq\ 1$, thus we get

$$0\ <\ h(n)\ \leq\ h(1)\ = \frac{(p+1)_1\ (p+1-\eta+\delta)_1}{(p+1-\eta)_1 (p+1+\alpha+\delta)_1} = \frac{(p+1)\ (p+1-\eta+\delta)}{(p+1-\eta)\ (p+1+\alpha+\delta)} \quad (6.11)$$

now, by using equation (2.1) and (6.11) in (6.9), then we get

$$|H(z)| \geq |z|^p - h(1)|z|^{1+p} \sum_{n=1}^{\infty} a_{n+p}, n\ \geq 1$$
$$|H(z)|\ \geq\ |z|^p \left(1 - \frac{(p+1)\ (p-\beta)(p+1-\eta+\delta)}{(p+1-\eta)(p+1+\alpha+\delta\ )(p+1+\alpha-\beta)\ a} \frac{\Gamma(c+k)}{\Gamma(b+k)} \frac{\Gamma(b)}{\Gamma(\mathrm{c})} |z|\right),$$

and

$$|H(z)|\ \leq\ |z|^p + h(1)|z|^{1+p} \sum_{n=1}^{\infty} a_{n+p}\ , n\ \geq\ 1,$$
$$|H(z)|\ \leq\ |z|^p \left(1 + \frac{(p+1)\ (p-\beta)(p+1-\eta+\delta)}{(p+1-\eta)(p+1+\alpha+\delta\ )(p+1+\alpha-\beta)\ a} \frac{\Gamma(c+k)}{\Gamma(b+k)} \frac{\Gamma(b)}{\Gamma(\mathrm{c})} |z|\right),$$

therefore,

$$\frac{\Gamma(p+1)\,\Gamma(p-\eta+\delta+1)}{\Gamma(p-\eta+1)\ \Gamma(p+\alpha+\delta+1)} |z|^{p-\eta} \left(1 - \frac{(p+1)\ (p-\beta)(p+1-\eta+\delta)}{(p+1-\eta)(p+1+\alpha+\delta\ )(p+1+\alpha-\beta)\ a} \frac{\Gamma(c+k)}{\Gamma(b+k)} \frac{\Gamma(b)}{\Gamma(\mathrm{c})} |z|\right)$$
$$\leq \left|I_{0,z}^{\alpha,\eta,\delta} f(z)\right| \leq \frac{\Gamma(p+1)\,\Gamma(p-\eta+\delta+1)}{\Gamma(p-\eta+1)\ \Gamma(p+\alpha+\delta+1)} |z|^{p-\eta} \left(1 + \frac{(p+1)\ (p-\beta)(p+1-\eta+\delta)}{(p+1-\eta)(p+1+\alpha+\delta\ )(p+1+\alpha-\beta)} \frac{\Gamma(c+k)}{a\Gamma(b+k)} \frac{\Gamma(b)}{\Gamma(\mathrm{c})} |z|\right).$$

Theorem is completely proved.

Corollary 2. If $\eta = -\alpha = -\lambda$ in theorem 6.1 and let the function $f(z)$ defined by (1.7) be in the class $A_p^k(\alpha,\beta)$. Then we have

$$\left|D_z^{-\lambda} f(z)\right| \geq \frac{\Gamma(p+1)}{\Gamma(p+\lambda+1)} |z|^{p+\lambda} \left(1 - \frac{(p+1)\ (p-\beta)}{(p+1+\lambda)(p+1+\lambda-\beta)\ a} \frac{\Gamma(c+k)}{\Gamma(b+k)} \frac{\Gamma(b)}{\Gamma(\mathrm{c})} |z|\right) \quad (6.12)$$

and

$$\left|D_z^{-\lambda} f(z)\right|\ \leq\ \frac{\Gamma(p+1)}{\Gamma(p+\lambda+1)} |z|^{p+\lambda} \left(\ 1 + \frac{(p+1)\ (p-\beta)}{(p+1+\lambda)(p+1+\lambda-\beta)} \frac{\Gamma(c+k)}{a\Gamma(b+k)} \frac{\Gamma(b)}{\Gamma(\mathrm{c})} |z|\right). \quad (6.13)$$

Corollary 3. If $\eta = -\alpha = \lambda$ in theorem 6.1 and let the function $f(z)$ defined by (1.7) be in the class $A_p^k(\alpha,\beta)$. Then we have

$$\left|D_z^{\lambda} f(z)\right|\ \geq\ \frac{\Gamma(p+1)}{\Gamma(p-\lambda+1)}\ |z|^{p-\lambda} \left(1 - \frac{(p+1)\ (p-\beta)}{(p+1-\lambda)(p+1-\lambda-\beta)a} \frac{\Gamma(c+k)}{\Gamma(b+k)} \frac{\Gamma(b)}{\Gamma(\mathrm{c})} |z|\right) \quad (6.14)$$

and

$$\left|D_z^{\lambda} f(z)\right|\ \leq\ \frac{\Gamma(p+1)}{\Gamma(p-\lambda+1)} |z|^{p-\lambda} \left(1 + \frac{(p+1)\ (p-\beta)}{(p+1-\lambda)(p+1-\lambda-\beta)\ a} \frac{\Gamma(c+k)}{\Gamma(b+k)} \frac{\Gamma(b)}{\Gamma(\mathrm{c})} |z|\right). \quad (6.15)$$

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