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Variational Formulation For Linear and nonlinear Problems

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Abstract: *This paper introduced the method of variational formulation for linear and nonlinear problems, and we show This method is suitable for equations of fluid dynamics since it is applicable to nonlinear evolutionary types of equations. We show that the variational characterization of a problem depends on the form of the equation underlying the problem.*
Keyword: Gateaux Derivatives, Symmetry, Variational Problem.

I. Introduction:

The problem of giving variational formulation for all linear or nonlinear problems has so far eluded solution. As a matter of fact,variational principle do not exist for many heat and mass transfer problems of intrest. We examine the question of existence of variational principle in a more systematic way. We first introduce the notion of Gateaux differentials and Gateaux derivatives in order to be able to give a general treatment of variational formulation for nonlinear differential equation.

1-Gateaux Derivatives:

Consider a vector field v,we derive v from a potential ,if $\nabla \times v = 0$,then v can be represented as the gradient of a potential,$v = \nabla\varphi$.It is clear from this elementary concept in vector calculus that if we regard the Euler equation in variational principle as the gradient of a functional ,analogous to a potential ,then we should not expect every differential equation to be derivable from a potential .We express in the general form

$$N(u) = \phi_v \qquad (1)$$

Where N denotes an operator which may by nonlinear. The set of elements u that satisfies the given initial or boundary conditions and the given functional class is called the domain of operator and is denoted by $D(N)$ which can be considered as a subset of a vector space U .The set of elements $v - N(u)$ constitutes the range of N denoted by $R(N)$.Suppose limit below exist

$$N'_u(\phi) = \lim_{\varepsilon \to 0} \frac{N(u+\varepsilon\phi) - N(u)}{\varepsilon} = \frac{\partial}{\partial \varepsilon} N(u+\varepsilon\phi)\Big|_{\varepsilon=0} \qquad (2)$$

Equation (2) is defined by the topology of the $V - space$ .The N'_u is called the Gateaux differential of the operator N in the direction ϕ and N'_u is called Gateaux derivative of the operator N .Now we define a bilinear functional $\langle v,u \rangle$,which is map $B : V \times U \to \Re$ .The expression $\langle v,u \rangle$ satisfies the following requirement :

i. It must be real-valued even if U and V are vector space over the complex number field.
ii. It must be bilinear over the real number field.
iii. It must be non-degenerate this implies that
   If $\langle v,u_0 \rangle = 0$,for every $v \in V$ then $u_0 = \phi_u$;
   If $\langle v_0,u \rangle = 0$,for every $u \in U$,then $v_0 = \phi_v$

The real number $S = \langle u, v \rangle$ is called the scalar product of the element $v \in V$ and $u \in U$. The V -space is called the dual space or the conjugate of the V-space or $V = U^*$.In any physical problem we take the bilinear functions such as

$$\langle v, u \rangle = \int \mathbf{V}(\mathbf{X}) \cdot \mathbf{U}(\mathbf{X}) \, d\Omega \qquad (3)$$

Where Ω is a subset of R^N and **X** is a point of Ω with **X** = $(x_1, x_2, ..., x_n)$

Here **V**.**U** is the scalar- product.

2-Symmetry condition:

The symmetry condition defined by the following:

$$\int \Psi N'_u \phi dV = \int \phi N'_u \Psi dV \qquad (4)$$

Where u, v and $\Psi \in X$ the operator N'_u is symmetric. The fact that (4) is the condition for the existence of function having the operator $N(u)$.

Theorem:

Suppose that the following conditions are satisfied:

i. N is an operator from a normal space E into its conjugate space E^*.

ii. N has a linear Gateaux differential $DN(x, h)$ at any every point of all balls $S : \|x - x_0\| < r$.

iii. The bilinear functional $(DN(x, h_1), h_2)$ be symmetric for every x in S

$$(DN(x, h_1), h_2) = (DN(x, h_2), h_1) \qquad (5)$$

for every h_1 and h_2 in E and every x in D. equation (5) is just the symmetry condition(4) if F is the functional arising in a variational principle ,then we deduce $N(u) = 0$ is the corresponding Euler equation, consequently ,the equation as whether a variational principle exist for given operator N depends on whether the operator has symmetric Gateaux differential expressed by the condition (4),to illustrate this notion we give the following example:

Example:

Let $f(u, u_j, u_{jk}) = 0$

Now from the formula (2) we deduce

$$N'_u = \left[\frac{\partial}{\partial \varepsilon} f(u + \varepsilon\phi, u_j + \varepsilon\phi_j, u_{jk} + \varepsilon\phi_{jk}) \right]_{\varepsilon=0}$$

$$= \frac{\partial f}{\partial u}\phi + \frac{\partial f}{\partial u_j}\phi_j + \frac{\partial f}{\partial u_{jk}} + \phi_{jk} \qquad (6)$$

To test the symmetry requirement ,we now integrate the following by parts

$$\int \Psi N'_u \phi dV = \int \Psi \left[\frac{\partial f}{\partial u}\phi + \frac{\partial f}{\partial u_j}\phi_j + \frac{\partial f}{\partial u_{jk}}\phi_{jk} \right] dV$$

$$\int \phi \left\{ \left[\frac{\partial f}{\partial u} - \nabla_j \left(\frac{\partial f}{\partial u_j} \right) + \nabla_k \nabla_j \left(\frac{\partial f}{\partial u_{jk}} \right) \right] \Psi + \left[-\frac{\partial f}{\partial u_j} + 2\nabla_k \left(\frac{\partial f}{\partial u_{jk}} \right) \right] \nabla_j \Psi + \frac{\partial f}{\partial u_{jk}} \nabla_k \nabla_j \Psi \right\} dV +$$

boundary term

$$\int \phi \overline{N}'_u \Psi dV + \text{ boundary terms.} \qquad (7)$$

Equation (7) defines the Gateaux derivatives $\overline{N'_u}$ .which maybe regarded as the ad joint to N_u .It should be noted that an ad joint is generally defined for a linear operator but the notion of an ad joint is also usefull for non linear operator.

3-Variational Formulation

3-1 variational formulation in the restricted sense :

Let a problem $N(u) = \phi_u$ with the operator N (which has be non linear $D(N) \subset U \to V = U^*$,the functional F given by:

$$\delta F = \langle N(u), \delta u \rangle$$

This means that the solutions to the problem are the critical points of F and vice versa ,the operator N : $D(u) \subset U \to R(N) \subset V = U^*$ be the gradient of the functional and $N'_u(u,\cdot)$ must be symmetric .

3-2 Variational Problem in the Extended Sense:

Let a problem $N(u) = \phi_v$,with $N : D(N) \subset U \to V = U^*$ determine a functional $\overline{F}$,if any ,whose critical points are solutions to the problem and vice versa. This means that for a given operator N ,there exist an operator $\overline{N}$ such that

$$\delta\overline{F} = \langle \overline{N}(u), \delta(u) \rangle$$

And the problems $N(u) = \phi_v$ and $\overline{N}(u) = \phi_v$ have the same solutions.

Theorem:

Consider the system

$$N(u) = \phi_v \qquad (8)$$

Where N is a nonlinear operator : $D(N) \subset U \to R(N) \subset U^*$ such that :

i. The solution of the problem exist.
ii. It is a unique.
iii. $D(N)$ is simply connected.
iv. The Gateaux derivatives $N'_u(u,\cdot)$exist.
v. $D(N'_u)$ is dense in U .
vi. $N'^*_u(u)$ is invertible for every $u \in D(N)$.

For every operator K that satisfies the condition (vii)

vii. $D(K) \supset R(N)$.
viii. $R(K) \subset D(N'^*_u)$
ix. It is linear .
x. It is invertible.
xi. It is symmetric the operator $\overline{N}$ defined by

$$\overline{N}(u) = N'^*_u(u, KN(u)) \qquad (9)$$

It is clear that the solution of (8) is the critical point of the functional

$$\overline{F}_1[u] = \frac{1}{2}\langle N(u), KN(u) \rangle \qquad (10)$$

Whose gradient is the $\overline{N}$ .The functional vanishes when the solution is attained further. If K is positive definite ,then $\overline{F}_1[u]$ is the minimum at the critical point. Now for every $v \in D(K)$ and $v \neq \phi_v$ since $\overline{F}_1[u_0] = 0$, it follows that $\overline{F}_1$ is minimum at u_0 .Note that the linear operator

$$R(u,v) = N'^*_u(u, Kv) \qquad (11)$$

Transforms a given operator N into a potential operator $\overline{N}(u) = R(u, Nu)$,and hence can be regarded as an integrating operator.

Example:

Consider the first –order nonlinear differential equation with the initial condition given by:

$$\dot{u}(t) = f(t, u(t)) \quad , \quad u(0) = a \quad , \quad u \in C^1(0, T) \qquad (12)$$

Where f is a prescribed function. We obtain a variational problem.

Solution:

This is a Cauchy problem so that its existence and uniqueness are guaranteed under the usual hypothesis. Here we have ,on comparing with (8) the relation

$$N(u) = \left\{ \frac{du(t)}{dt} - f(t, u(t)); u(0) = a; u \in C^1(0, T) \right\} \qquad (13)$$

$$N'_u(\phi) = \left\{ \frac{d}{dt}\phi(t) - \frac{\partial f}{\partial u}\phi(t); \phi(0) = 0; \phi \in C^1(0, T) \right\} \qquad (14)$$

$$N'^*_u \Psi = \left\{ -\frac{d}{dt}\Psi(t) - \frac{\partial f}{\partial u}\Psi(t); \Psi(T) = 0; \Psi \in A(0, T) \right\} \qquad (15)$$

The adjoint homogeneous problem is

$$-\frac{d}{dt}\Psi(t) - \frac{\partial f}{\partial u}\Psi(t) = 0 \ , \quad \Psi(T) = 0 \qquad (16)$$

This equation is linear in Ψ with variable coefficients containing $u(t)$.It can be easily shown that it has only null solution and so the operator N'^*_u is invertible. Thus the conditions of above theorem are fulfilled. Let us introduce the function h as follows

$$h(t, u(t), \dot{u}(t)) = \dot{u}(t) - f(t, u(t)). \qquad (17)$$

We know that the operator K appearing in the integral operator (11) should be invertible, symmetric with a range contained in $D(N'^*_u)$.These integral operator whose kernel is Green's function of such a differential operator .Take ,for instance, the operator

$$L = \left\{ -\frac{d^2}{dt^2}, \dot{u}(0) = 0, u(T) = 0, u \in C^2(0, T) \right\} \qquad (18)$$

Which is clearly symmetric and positive –definite and invertible. Its inverse is

$$Kv = \int_0^T [-(t-\tau)H(t-\tau) + (T-\tau)]v\,d\tau \qquad (19)$$

Where $H(t)$ is Heaviside unit step function with the Green's function

$$g(t, \tau) = -(t-\tau)H(t-\tau) + (T-\tau) \qquad (20)$$

Using (10) ,we now have the variational formulation of the problem (12) with the required functional given by

$$\overline{F}_1(u) = \frac{1}{2}\int_0^T h(t, u(t), \dot{u}(t)) \int_0^T g(t, \tau) h(t, u(\tau), \dot{u}(\tau))\, d\tau dt \qquad (21)$$

Where h and g are given by (16) and (20) . Further using (11) the integrating operator $R(u, v)$ is given by

$$R(u,v)=\left[-\frac{d}{dt}-\frac{\partial f}{\partial u}\right]\int_0^T\left[-(t-\tau)H(t-\tau)+(T-\tau)\right]v(\tau)d\tau$$

Which after integrating by parts, reduce to

$$R(u,v)=\int_0^T v(\tau)d\tau-\frac{\partial f}{\partial u}\cdot\int_0^T\left[-(t-\tau)H(t-\tau)+(T-\tau)\right]v(\tau)d\tau$$

If we use Green's function for the operator

$$L=\left\{-\frac{d^2}{dt^2},\dot{u}(0)=0,u(T)=0,u\in C^2(0,T)\right\}$$

Instead of (18) in (21) and perform two integrations by parts, we obtain the functional

$$\overline{F}_1(u)=\frac{1}{2}\int_0^T u(t)\int_0^T\frac{\partial^2 g}{\partial\tau\partial t}u(\tau)d\tau dt+\int_0^T f(t,u(t))\int_0^T\frac{\partial g}{\partial\tau}u(\tau)d\tau dt+\frac{1}{2}\int_0^T f(t,u(t))\int_o^T g(t,\tau)f(\tau,u(\tau))d\tau dt \quad (22)$$

Where $g(t,\tau)$ is given by

$$g(t,\tau)=-(t-\tau)H(t-\tau)+(T-\tau)\frac{t}{T} \quad (23)$$

It is interesting to note that $\overline{F}_1(u)$ in (22) is free from $\dot{u}(t)$ which is absorbed by the kernel $g(t,\tau)$.

Conclusion:

1- This method suitable for equations of fluid dynamics since it is applicable to nonlinear evolutionary types of equations.
2- This method provide us a procedure for obtaining the required functional for the both homogeneous and nonhomogeneous boundary conditions
3- We show that the variational characterization of a problem depends on the form of the equation underlying the problem.

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